Investigating Recursive Rules

1. Find the first five terms of each sequence.
   a. \( a_1 = 3 \)
      \[ a_n = a_{n-1} + 5 \]
   b. \( a_1 = 3 \)
      \[ a_n = 2a_{n-1} \]

2. Based on the lists of terms you found in Step 1, what type of sequence is the sequence in part (a)? in part (b)?
EXAMPLE 2  Writing a Recursive Rule for an Arithmetic Sequence

Write the indicated rule for the arithmetic sequence with \(a_1 = 4\) and \(d = 3\).

\(\text{a. an explicit rule} \quad \text{b. a recursive rule}\)

**SOLUTION**

\(\text{a. From Lesson 11.2 you know that an explicit rule for the } n\text{th term of the}\)

\(\text{arithmetic sequence is:}\)

\[a_n = a_1 + (n - 1)d\]

\[= 4 + (n - 1)3\]

\[= 1 + 3n\]

\text{Simplify.}

\(\text{b. To find the recursive equation, use the fact that you can obtain } a_n\)

\(\text{by adding the common difference } d\) to the previous term.

\[a_n = a_{n-1} + d\]

\[= a_{n-1} + 3\]

\text{Substitute for } d.

A recursive rule for the sequence is \(a_1 = 4, a_n = a_{n-1} + 3\).

EXAMPLE 3  Writing a Recursive Rule for a Geometric Sequence

Write the indicated rule for the geometric sequence with \(a_1 = 3\) and \(r = 0.1\).

\(\text{a. an explicit rule} \quad \text{b. a recursive rule}\)

**SOLUTION**

\(\text{a. From Lesson 11.3 you know that an explicit rule for the } n\text{th term of the}\)

\(\text{geometric sequence is:}\)

\[a_n = a_1 r^{n-1}\]

\[= 3(0.1)^{n-1}\]

\text{Substitute for } a_1 \text{ and } r.

\(\text{b. To write a recursive rule, use the fact that you can obtain } a_n\)

\(\text{by multiplying the previous term by } r.\)

\[a_n = r \cdot a_{n-1}\]

\[= (0.1)a_{n-1}\]

\text{Substitute for } r.

A recursive rule for the sequence is \(a_1 = 3, a_n = (0.1)a_{n-1}\).

EXAMPLE 4  Writing a Recursive Rule

Write a recursive rule for the sequence 1, 2, 2, 4, 8, 32, . . . .

**SOLUTION**

Beginning with the third term in the sequence, each term is the product of the two
previous terms. Therefore, a recursive rule is given by:

\[a_1 = 1, a_2 = 2, a_n = a_{n-2} \cdot a_{n-1}\]
GOAL 2 USING RECURSIVE RULES IN REAL LIFE

EXAMPLE 5 Using a Recursive Rule

**FISH** A lake initially contains 5200 fish. Each year the population declines 30% due to fishing and other causes, and the lake is restocked with 400 fish.

a. Write a recursive rule for the number \(a_n\) of fish at the beginning of the \(n\)th year. How many fish are in the lake at the beginning of the fifth year?

b. What happens to the population of fish in the lake over time?

**Solution**

a. Because the population declines 30% each year, 70% of the fish remain in the lake from one year to the next, and new fish are added.

**VERBAL MODEL**

Fish at start of \(n\)th year = 0.7 Fish at start of \((n - 1)\)st year + New fish added

**LABELS**

Fish at start of \(n\)th year = \(a_n\)

Fish at start of \((n - 1)\)st year = \(a_{n-1}\)

New fish added = 400

**ALGEBRAIC MODEL**

\[ a_n = (0.7)a_{n-1} + 400 \]

A recursive rule is:

\[ a_1 = 5200, \ a_n = (0.7)a_{n-1} + 400 \]

You can use a graphing calculator to find \(a_5\), the number of fish in the lake at the beginning of the fifth year. Enter the number of fish at the beginning of the first year, which is \(a_1 = 5200\). Then enter the rule \(0.7 \times \text{Ans} + 400\) to find \(a_2\). Press \(\text{ENTER}\) three more times to find \(a_5 \approx 2262\).

- There are about 2262 fish in the lake at the beginning of the fifth year.

b. To determine what happens to the lake’s fish population over time, continue pressing \(\text{ENTER}\) on the calculator. The calculator screen at the right shows the fish populations for years 44–50. Observe that the numbers approach about 1333.

- Over time, the population of fish in the lake stabilizes at about 1333 fish.
1. Complete this statement: The expression \( \prod \) represents the product of all integers from 1 to \( n \).

2. Explain the difference between an explicit rule for a sequence and a recursive rule for a sequence.

3. Give an example of an explicit rule for a sequence and a recursive rule for a sequence.

Write the first five terms of the sequence.

4. \( a_1 = 1 \)
\( a_n = a_{n-1} + 1 \)
5. \( a_1 = 2 \)
\( a_n = 4a_{n-1} - 1 \)
6. \( a_0 = 1 \)
\( a_n = a_{n-1} - 2 \)

7. \( a_1 = -1 \)
\( a_n = -3a_{n-1} \)
8. \( a_1 = 2 \)
\( a_n = 2a_{n-1} - 3 \)
9. \( a_0 = 3 \)
\( a_n = (a_{n-1})^2 + 1 \)

Write a recursive rule for the sequence.

10. 21, 17, 13, 9, 5, . . .
11. 2, 6, 18, 54, 162, . . .
12. \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \)

13. **Fish** Suppose each year the lake in Example 5 is restocked with 750 fish.
How many fish are in the lake at the beginning of the fifth year?

**WRITING TERMS** Write the first five terms of the sequence.

14. \( a_0 = 1 \)
\( a_n = a_{n-1} + 4 \)
15. \( a_1 = 4 \)
\( a_n = n + a_{n-1} + 6 \)
16. \( a_0 = 0 \)
\( a_n = a_{n-1} - n^2 \)

17. \( a_0 = -4 \)
\( a_n = a_{n-1} - 8 \)
18. \( a_1 = 2 \)
\( a_n = (a_{n-1})^2 + 2 \)
19. \( a_0 = 5 \)
\( a_n = n^2 - a_{n-1} \)

20. \( a_1 = 10 \)
\( a_n = 3a_{n-1} \)
21. \( a_0 = 2 \)
\( a_n = n^2 + 2n - a_{n-1} \)
22. \( a_0 = 3 \)
\( a_n = (a_{n-1})^2 - 2 \)

23. \( a_0 = 48 \)
\( a_n = \frac{1}{2}a_{n-1} + 2 \)
24. \( a_0 = 4, a_1 = 2 \)
\( a_n = a_{n-1} - a_{n-2} \)
25. \( a_0 = 1, a_2 = 3 \)
\( a_n = a_{n-1} - a_{n-2} \)

**WRITING RULES** Write an explicit rule and a recursive rule for the sequence.
(Recall that \( d \) is the common difference of an arithmetic sequence and \( r \) is the common ratio of a geometric sequence.)

26. \( a_1 = 2 \)
\( r = 10 \)
27. \( a_1 = 3 \)
\( d = 10 \)
28. \( a_1 = 10 \)
\( r = 2 \)

29. \( a_1 = 5 \)
\( d = 3 \)
30. \( a_1 = 0 \)
\( d = -1 \)
31. \( a_1 = 5 \)
\( r = 2.5 \)

32. \( a_1 = 14 \)
\( d = \frac{1}{2} \)
33. \( a_1 = \frac{1}{2} \)
\( r = 4 \)
34. \( a_1 = -1 \)
\( d = -\frac{3}{2} \)
11.5 Recursive Rules for Sequences

**WRITING RULES** Write a recursive rule for the sequence. The sequence may be arithmetic, geometric, or neither.

35. 1, 7, 13, 19, . . . 36. 66, 33, 16.5, 8.25, . . . 37. 41, 32, 23, 14, . . .
38. 3, 8, 63, 3968, . . . 39. 33, 11, \frac{11}{3}, \frac{11}{9}, . . . 40. 7.2, 3.2, −0.8, −4.8, . . .
41. 2, 5, 10, 50, 500, . . . 42. 6, 6\sqrt{2}, 12, 12\sqrt{2}, . . . 43. 48, 4.8, 0.48, 0.048, . . .

44. **ON LAYAWAY** Suppose you buy a $500 camcorder on layaway by making a down payment of $150 and then paying $25 per month. Write a recursive rule for the total amount of money paid on the camcorder at the beginning of the \(n\)th month. How much will you have left to pay on the camcorder at the beginning of the twelfth month?

**FRAC TAL TREE** In Exercises 45 and 46, use the following information.
A fractal tree starts with a single branch (the trunk). At each stage the new branches from the previous stage each grow two more branches, as shown.

45. List the number of new branches in each of the first seven stages. What type of sequence do these numbers form?
46. Write an explicit rule and a recursive rule for the sequence in Exercise 45.

**POOL CARE** In Exercises 47 and 48, use the following information.
You have just bought a new swimming pool and need to add chlorine to the water. You add 32 ounces of chlorine the first week and 14 ounces every week thereafter. Each week 40% of the chlorine in the pool evaporates.

47. Write a recursive rule for the amount of chlorine in the pool each week. How much chlorine is in the pool at the beginning of the sixth week?
48. What happens to the amount of chlorine after an extended period of time?

**TREE FARM** In Exercises 49 and 50, use the following information.
Suppose a tree farm initially has 9000 trees. Each year 10% of the trees are harvested and 800 seedlings are planted.

49. Write a recursive rule for the number of trees on the tree farm at the beginning of the \(n\)th year. How many trees remain at the beginning of the fourth year?
50. What happens to the number of trees after an extended period of time?

**DOSAGE** In Exercises 51–54, use the following information.
A person repeatedly takes 20 milligrams of a prescribed drug every four hours. Suppose that 30% of the drug is removed from the bloodstream every four hours.

51. Write a recursive rule for the amount of the drug in the bloodstream after \(n\) doses.
52. What value does the drug level in the person’s body approach after an extended period of time? This value is called the maintenance level.
53. Suppose the first dosage is doubled (to 40 milligrams), but the normal dosage is taken thereafter. Does the maintenance level from Exercise 52 change?
54. Suppose every dosage is doubled. Does the maintenance level double as well?
55. **CRITICAL THINKING** Give an example of a sequence in which each term after the third term is a function of the three terms preceding it. Write a recursive rule for the sequence and find the first 8 terms.

56. **MULTIPLE CHOICE** What is the fifth term of the sequence whose first term is \(a_1 = 10\) and whose \(n\)th term is \(a_n = 2a_{n-1} + 9\)?

- A) 67
- B) 143
- C) 286
- D) 295
- E) 599

57. **MULTIPLE CHOICE** What is a recursive equation for the sequence 4, -6.6, 10.89, -17.9685, . . .?

- A) \(a_n = (–2.6)a_{n-1}\)
- B) \(a_n = (–1.65)a_{n-1}\)
- C) \(a_n = (2.6)a_{n-1}\)
- D) \(a_n = (1.65)a_{n-1}\)

58. **PIECEWISE-DEFINED SEQUENCE** You can define a sequence using a piecewise rule. The following is an example of a piecewise-defined sequence.

\[
a_1 = 7, \quad a_n = \begin{cases} \frac{a_{n-1}}{2}, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1, & \text{if } a_{n-1} \text{ is odd} \end{cases}
\]

a. Write the first ten terms of the sequence.

b. **LOGICAL REASONING** Choose three different values for \(a_1\) (other than \(a_1 = 7\)). For each value of \(a_1\), find the first ten terms of the sequence. What conclusions can you make about the behavior of this sequence?

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### MIXED REVIEW

**EVALUATING POWERS** Evaluate the power. (Review 1.2 for 12.1)

- 59. \(2^5\)
- 60. \(6^4\)
- 61. \(8^4\)
- 62. \(12^3\)

- 63. \(26^3\)
- 64. \(10^5\)
- 65. \(18^3\)
- 66. \(3^7\)

**OPERATIONS WITH RATIONAL EXPRESSIONS** Perform the indicated operation and simplify. (Review 9.5)

- 67. \(\frac{3}{5x} + \frac{3}{7x}\)
- 68. \(-\frac{2}{7x} - \frac{5}{3x}\)
- 69. \(\frac{x+1}{x^2-9} - \frac{5}{x-3}\)

- 70. \(\frac{2x^2}{3x+5} - \frac{14}{x+7}\)
- 71. \(\frac{4x+1}{x^2-4} - \frac{3}{x-2}\)
- 72. \(\frac{x^2-1}{x+2} - \frac{3}{x+1}\)

**FINDING POINTS OF INTERSECTION** Find the points of intersection, if any, of the graphs in the system. (Review 10.7)

- 73. \(x^2 + y^2 = 4, \quad 2x + y = -1\)
- 74. \(x^2 + y^2 = 25, \quad y = x - 1\)
- 75. \(x^2 + 4y^2 = 16, \quad y = 3x + 1\)

- 76. \(x^2 + y^2 = 10, \quad 4x + y = 6\)
- 77. \(x^2 + y^2 = 30, \quad y = x + 2\)
- 78. \(16x^2 + y^2 = 32, \quad \frac{1}{4}x - \frac{1}{2}y = 2\)

**WRITING TERMS** Write the first six terms of the sequence. (Review 11.1)

- 79. \(a_n = 8 - n\)
- 80. \(a_n = n^4\)
- 81. \(a_n = n^2 + 9\)

- 82. \(a_n = (n+3)^2\)
- 83. \(a_n = \frac{n}{n + 4}\)
- 84. \(a_n = \frac{n + 3}{n + 1}\)
Quiz 2

Find the sum of the infinite geometric series if it has one. (Lesson 11.4)

1. \(\sum_{n=0}^{\infty} 4 \left(\frac{1}{9}\right)^n\)
2. \(\sum_{n=1}^{\infty} 5 \left(-\frac{6}{7}\right)^{n-1}\)
3. \(\sum_{n=0}^{\infty} -3 \left(\frac{4}{7}\right)^n\)
4. \(\sum_{n=0}^{\infty} \frac{4}{3} \left(\frac{5}{7}\right)^n\)

Find the common ratio of the infinite geometric series with the given sum and first term. (Lesson 11.4)

5. \(S = 5, a_1 = 1\)
6. \(S = 12, a_1 = 1\)
7. \(S = 24, a_1 = 3\)

Write the repeating decimal as a fraction. (Lesson 11.4)

8. 0.888...
9. 0.1515...
10. 126.126126...

Write the first five terms of the sequence. (Lesson 11.5)

11. \(a_1 = 5\)
\(a_n = a_{n-1} + 3\)
12. \(a_0 = 1\)
\(a_n = 4a_{n-1}\)
13. \(a_1 = 17\)
\(a_n = a_{n-1} + n\)
14. \(a_1 = 1, a_2 = 2\)
\(a_n = a_{n-1} - a_{n-2}\)
15. \(a_1 = 2, a_2 = 4\)
\(a_n = a_{n-1} \cdot a_{n-2}\)
16. \(a_1 = 10, a_2 = 10\)
\(a_n = a_{n-2} + a_{n-1}\)
17. BALL BOUNCE You drop a ball from a height of 8 feet. Each time it hits the ground, it bounces 40% of its previous height. Find the total distance traveled by the ball. (Lesson 11.4)

The Fibonacci Sequence

IN 1202 the mathematician Leonardo Fibonacci wrote Liber Abaci in which he proposed the following rabbit problem.

Begin with a pair of newborn rabbits that never die. When a pair of rabbits is two months old, it begins producing a new pair of rabbits each month.

This problem can be represented by a sequence, known as the Fibonacci sequence. The numbers that make up the sequence are called Fibonacci numbers. The ratio of two Fibonacci numbers approximates the same number, denoted by \(\Phi\). The Greeks called this number the golden ratio.

1. Draw a tree diagram to illustrate the sequence.
2. If the initial pair of rabbits produces their first pair of rabbits in January, how many pairs of rabbits will there be in December of that year? What happens to the rabbit population over time?

TODAY we know that Fibonacci numbers occur in nature, such as in the spiral patterns on the head of a sunflower or the surface of a pineapple.

APPLICATION LINK
www.mcdougallittell.com

Month | 1 | 2 | 3 | 4 | 5 | 6 |... |
Pairs at start of month | 1 | 1 | 2 | 3 | 5 | 8 |... |

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