Geometric Sequences and Series

GOAL 1 Using Geometric Sequences and Series

In a geometric sequence, the ratio of any term to the previous term is constant. This constant ratio is called the common ratio and is denoted by \( r \).

Example 1 Identifying Geometric Sequences

Decide whether each sequence is geometric.

a. 1, 2, 6, 24, 120, . . .

b. 81, 27, 9, 3, 1, . . .

Solution

To decide whether a sequence is geometric, find the ratios of consecutive terms.

a. \[
\frac{a_2}{a_1} = \frac{2}{1} = 2, \quad \frac{a_3}{a_2} = \frac{6}{2} = 3, \quad \frac{a_4}{a_3} = \frac{24}{6} = 4, \quad \frac{a_5}{a_4} = \frac{120}{24} = 5
\]

The ratios are different, so the sequence is not geometric.

b. \[
\frac{a_2}{a_1} = \frac{27}{81} = \frac{1}{3}, \quad \frac{a_3}{a_2} = \frac{9}{27} = \frac{1}{3}, \quad \frac{a_4}{a_3} = \frac{3}{9} = \frac{1}{3}, \quad \frac{a_5}{a_4} = \frac{1}{3}
\]

The ratios are the same, so the sequence is geometric.

Rule for a Geometric Sequence

The \( n \)th term of a geometric sequence with first term \( a_1 \) and common ratio \( r \) is given by:

\[
a_n = a_1r^{n-1}
\]

Example 2 Writing a Rule for the \( n \)th Term

Write a rule for the \( n \)th term of the sequence \(-8, -12, -18, -27, . . .\). Then find \( a_8 \).

Solution

The sequence is geometric with first term \( a_1 = -8 \) and common ratio \( r = \frac{-12}{-8} = \frac{3}{2} \).

So, a rule for the \( n \)th term is:

\[
a_n = a_1r^{n-1}
\]

Write general rule.

\[
= -8 \left( \frac{3}{2} \right)^{n-1}
\]

Substitute for \( a_1 \) and \( r \).

The 8th term is \( a_8 = -8 \left( \frac{3}{2} \right)^8 - 1 = -\frac{2187}{16} \).
EXAMPLE 3 Finding the nth Term Given a Term and the Common Ratio

One term of a geometric sequence is \( a_3 = 5 \). The common ratio is \( r = 2 \).

a. Write a rule for the \( n \)th term. 

**Solution**

a. Begin by finding the first term as follows.

\[
a_n = a_1 r^{n-1} \quad \text{Write general rule.}
\]

\[
a_3 = a_1 r^{3-1} \quad \text{Substitute 3 for } n.
\]

\[
a_3 = a_1 (2)^2 \quad \text{Substitute for } a_3 \text{ and } r.
\]

\[
5 = a_1 \cdot 4 \quad \text{Solve for } a_1.
\]

So, a rule for the \( n \)th term is:

\[
a_n = a_1 r^{n-1} \quad \text{Write general rule.}
\]

\[
= 1.25(2)^{n-1} \quad \text{Substitute for } a_1 \text{ and } r.
\]

b. The graph is shown at the right. Notice that the points lie on an exponential curve. This is true for any geometric sequence with \( r > 0 \).

EXAMPLE 4 Finding the nth Term Given Two Terms

Two terms of a geometric sequence are \( a_2 = 45 \) and \( a_5 = -1215 \). Find a rule for the \( n \)th term.

**Solution**

Write a system of equations using \( a_n = a_1 r^{n-1} \) and substituting 2 for \( n \) (Equation 1) and then 5 for \( n \) (Equation 2).

\[
a_2 = a_1 r^{2-1} \quad 45 = a_1 r \quad \text{Equation 1}
\]

\[
a_5 = a_1 r^{5-1} \quad -1215 = a_1 r^4 \quad \text{Equation 2}
\]

Solve the system.

\[
\frac{45}{r} = a_1 \quad \text{Solve Equation 1 for } a_1.
\]

\[
-1215 = \frac{45}{r} (r^4)
\]

\[
-1215 = 45r^3
\]

\[
-27 = r^3
\]

\[
-3 = r
\]

\[
45 = a_1 (-3)
\]

\[
-15 = a_1
\]

Find a rule for \( a_n \).

\[
a_n = a_1 r^{n-1} \quad \text{Write general rule.}
\]

\[
a_n = -15(-3)^{n-1} \quad \text{Substitute for } a_1 \text{ and } r.
\]

A rule for the \( n \)th term is \( a_n = -15(-3)^{n-1} \).
The expression formed by adding the terms of a geometric sequence is called a **geometric series**. As with an arithmetic series, the sum of the first \( n \) terms of a geometric series is denoted by \( S_n \). You can develop a rule for \( S_n \) as follows.

\[
S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1}
\]

\[
-rS_n = -a_1r - a_1r^2 - a_1r^3 - \cdots - a_1r^{n-1} - a_1r^n
\]

\[
S_n(1 - r) = a_1 - a_1r^n
\]

Therefore, \( S_n(1 - r) = a_1(1 - r^n) \). If \( r \neq 1 \), you can divide both sides of this equation by \( 1 - r \) to obtain the following rule for \( S_n \).

**THE SUM OF A FINITE GEOMETRIC SERIES**

The sum \( S_n \) of the first \( n \) terms of a geometric series with common ratio \( r \neq 1 \) is:

\[
S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)
\]

**EXAMPLE 5 Finding a Sum**

Consider the geometric series \( 1 + 5 + 25 + 125 + 625 + \cdots \).

a. Find the sum of the first 10 terms.

b. Find \( n \) such that \( S_n = 3906 \).

**SOLUTION**

a. To begin, notice that \( a_1 = 1 \) and \( r = 5 \). Therefore:

\[
S_{10} = a_1\left(\frac{1 - r^{10}}{1 - r}\right)
\]

Write rule for \( S_{10} \).

\[
= 1\left(\frac{1 - 5^{10}}{1 - 5}\right)
\]

Substitute for \( a_1 \) and \( r \).

\[
= 2,441,406
\]

Simplify.

The sum of the first 10 terms is 2,441,406.

b. \( a_1\left(\frac{1 - r^n}{1 - r}\right) = S_n \)

Write general rule.

\[
1\left(\frac{1 - 5^n}{1 - 5}\right) = 3906
\]

Substitute for \( a_1 \), \( r \), and \( S_n \).

\[
\frac{1 - 5^n}{-4} = 3906
\]

Simplify.

\[
1 - 5^n = -15,624
\]

Multiply each side by \(-4\).

\[
-5^n = -15,625
\]

Subtract 1 from each side.

\[
5^n = 15,625
\]

Divide each side by \(-1\).

\[
n = \log 15,625 / \log 5 = 6
\]

Solve for \( n \).

\[\] So, \( S_n = 3906 \) when \( n = 6 \).
EXAMPLE 6  Writing a Geometric Sequence

CELLULAR TELEPHONES  In 1990 the average monthly bill for cellular telephone service in the United States was $80.90. From 1990 through 1997, the average monthly bill decreased by about 8.6% per year. Source: Statistical Abstract of the United States

a. Write a rule for the average monthly cellular telephone bill $a_n$ (in dollars) in terms of the year. Let $n = 1$ represent 1990.

b. What was the average monthly cellular telephone bill in 1993?

c. When did the average monthly cellular telephone bill fall to $50$?

SOLUTION

a. Because the average monthly bill decreased by the same percent each year, the average monthly bills from year to year form a geometric sequence. Use $a_1 = 80.9$ and $r = 1 - 0.086 = 0.914$. A rule for the average monthly bill is:

$$a_n = 80.9(0.914)^{n-1}$$

b. In 1993, $n = 4$. So, the average monthly bill was

$$a_4 = 80.9(0.914)^3 ≈ 61.77$$

You want to find $n$ such that $a_n = 50$.

$$80.9(0.914)^{n-1} = 50$$

Divide each side by 80.9.

$$(0.914)^{n-1} = 0.618$$

Solve for $n - 1$.

$$n - 1 = \log_{0.914} 0.618$$

Solve for $n$.

$$n ≈ 6$$

The average monthly cellular telephone bill reached $50 in 1995 (when $n = 6$).

EXAMPLE 7  Finding the Sum of a Geometric Series

Use the model for the average monthly cellular telephone bill in Example 6. On average, what did a person pay for cellular telephone service during 1990–1997?

SOLUTION

Because the model $a_n = 80.9(0.914)^{n-1}$ gives the average monthly bill, the model $b_n = 12(80.9)(0.914)^{n-1} = 970.8(0.914)^{n-1}$ gives the average annual bill. Using $a_1 = 970.8$ and $r = 0.914$, you can estimate a person’s total cost for cellular telephone service during the 8 year period 1990–1997 to be:

$$S_8 = a_1 \left(1 - r^8 \right) \frac{1}{1 - r}$$

$$= 970.8 \left(1 - (0.914)^8 \right) \frac{1}{1 - 0.914}$$

$$≈ 5790$$

**Guided Practice**

1. Complete this statement: The constant ratio in a geometric sequence is called the \( ? \) ratio and is denoted by \( ? \).

2. What makes a sequence geometric?

3. State the rule for the sum of the first \( n \) terms of a geometric series.

Find the common ratio of the geometric sequence.

4. \( 4, 12, 36, 108, \ldots \)
5. \( 1, 6, 36, 216, \ldots \)
6. \( 2, -6, 18, -54, 162, \ldots \)

7. \( 7, 14, 28, 56, \ldots \)
8. \( 64, -32, 16, -8, \ldots \)
9. \( 10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots \)

Write the next term and find a rule for the \( n \)th term of the geometric sequence.

10. \( 1, 3, 9, 27, \ldots \)
11. \( 2, 8, 32, 128, \ldots \)
12. \( 1, -6, 36, -216, \ldots \)

13. \( 375, -75, 15, -3, \ldots \)
14. \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \)
15. \( -28, 14, -7, \frac{7}{2}, \frac{7}{4}, \ldots \)

Write a rule for the \( n \)th term of the geometric sequence.

16. \( r = 3, a_1 = 2 \)
17. \( r = -2, a_1 = 6 \)
18. \( r = -3, a_1 = 12 \)
19. \( a_1 = \frac{1}{2}, a_3 = 6 \)
20. \( a_2 = 5, a_4 = \frac{1}{5} \)
21. \( a_2 = 28, a_5 = -1792 \)

22. Find the sum of the first 8 terms of the geometric series \( 1 + 8 + 64 + 512 + \ldots \).

23. **Cellular Phones** Use the model from Example 6 to find the average monthly bill for cellular telephone service in 1997.

**Practice and Applications**

**Classifying Sequences** Decide whether the sequence is arithmetic, geometric, or neither. Explain your answer.

24. \( 6, 24, 96, 384, \ldots \)
25. \( 1, 3, 7, 13, \ldots \)
26. \( 4, 13, 22, 31, \ldots \)

27. \( 3, -1, -5, -9, \ldots \)
28. \( -11, -7, -3, 1, \ldots \)
29. \( \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \ldots \)

30. \( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots \)
31. \( -\frac{3}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{3}{32}, \ldots \)
32. \( -\frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \ldots \)

**Finding Common Ratios** Find the common ratio of the geometric sequence.

33. \( 1, 4, 16, 64, \ldots \)
34. \( 3, 6, 12, 24, \ldots \)
35. \( -3, 6, -12, 24, \ldots \)

36. \( 5, 40, 320, 2560, \ldots \)
37. \( 136, 68, 34, 17, \ldots \)
38. \( -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \ldots \)

**Writing Terms** Write a rule for the \( n \)th term of the geometric sequence. Then find \( a_n \).

39. \( 1, -4, 16, -64, \ldots \)
40. \( 5, 10, 20, 40, \ldots \)
41. \( 2, 14, 98, 686, \ldots \)

42. \( 6, -30, 150, -750, \ldots \)
43. \( 5, -\frac{5}{3}, \frac{5}{9}, -\frac{5}{27}, \ldots \)
44. \( 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots \)
WRITING RULES Write a rule for the \( n \)th term of the geometric sequence.

45. \( r = 3, a_1 = 4 \) \hspace{1cm} 46. \( r = \frac{1}{3}, a_1 = 45 \) \hspace{1cm} 47. \( r = 6, a_3 = 72 \)

48. \( r = \frac{1}{8}, a_1 = 4 \) \hspace{1cm} 49. \( r = 8, a_1 = -2 \) \hspace{1cm} 50. \( a_1 = -\frac{1}{2}, a_4 = -16 \)

51. \( a_3 = 10, a_6 = 300 \) \hspace{1cm} 52. \( a_2 = -20, a_4 = -5 \) \hspace{1cm} 53. \( a_2 = -30, a_5 = 3750 \)

GRAPHING SEQUENCES Graph the geometric sequence.

54. \( a_n = 4(2)^{n-1} \) \hspace{1cm} 55. \( a_n = 3(5)^{n-1} \) \hspace{1cm} 56. \( a_n = 2(3)^{n-1} \)

57. \( a_n = 8(3)^{n-1} \) \hspace{1cm} 58. \( a_n = 5\left(\frac{1}{2}\right)^{n-1} \) \hspace{1cm} 59. \( a_n = 4\left(\frac{3}{2}\right)^{n-1} \)

FINDING SUMS For part (a), find the sum of the first \( n \) terms of the geometric series. For part (b), find \( n \) for the given sum \( S_n \).

60. \( 1 + 4 + 16 + 64 + \cdots \) \hspace{1cm} 61. \( 1 + 9 + 81 + 729 + \cdots \)

\( a. \ n = 14 \) \hspace{1cm} \( b. \ S_n = 341 \) \hspace{1cm} \( a. \ n = 10 \) \hspace{1cm} \( b. \ S_n = 820 \)

62. \( 7 + (-21) + 63 + (-189) + \cdots \) \hspace{1cm} 63. \(-90 + 30 + (-10) + \frac{10}{3} + \cdots \)

\( a. \ n = 18 \) \hspace{1cm} \( b. \ S_n = 3829 \) \hspace{1cm} \( a. \ n = 16 \) \hspace{1cm} \( b. \ S_n = -66.67 \)

USING SUMMATION NOTATION Find the sum of the series.

64. \( \sum_{i=1}^{10} 6(2)^{i-1} \) \hspace{1cm} 65. \( \sum_{i=1}^{8} 5(4)^{i-1} \) \hspace{1cm} 66. \( \sum_{i=0}^{9} 12\left(-\frac{1}{2}\right)^{i} \)

67. \( \sum_{i=1}^{10} 8\left(\frac{3}{4}\right)^{i-1} \) \hspace{1cm} 68. \( \sum_{i=0}^{6} 4\left(\frac{3}{2}\right)^{i} \) \hspace{1cm} 69. \( \sum_{i=1}^{12} (-2)^{i-1} \)

TENNIS In Exercises 70 and 71, use the following information.
The men’s U.S. Open tennis tournament is held annually in Flushing Meadow in New York City. In the first round of the tournament, 64 matches are played. In each successive round, the number of matches played decreases by one half.

\( \text{Source: United States Tennis Association} \)

70. Find a rule for the number of matches played in the \( n \)th round. For what values of \( n \) does your rule make sense?

71. Find the total number of matches played in the men’s U.S. Open tennis tournament.

COMPUTER SCIENCE In Exercises 72 and 73, use the following information.

When a computer must find an item in an ordered list of data (such as an alphabetical list of names), it may be programmed to perform a binary search. This search technique involves jumping to the middle of the list and deciding whether the item is there. If not, the computer decides whether the item comes before or after the middle. Half of the list is then ignored on the next pass through the list, and the computer jumps to the middle of the remaining list. This is repeated until the item is found.

72. An ordered list contains 1024 items. Find a rule for the number of items remaining after the \( n \)th pass through the list.

73. In the worst case, the item to be found is the only one left in the list after \( n \) passes through the list. What is the worst-case value of \( n \) for a binary search of a list with 1024 items?
In Exercises 74–77, use the following information.

In 1990 factory sales of pagers in the United States totaled $118 million. From 1990 through 1996, the sales increased by about 20% per year.

74. Write a rule for pager sales \( a_n \) (in millions of dollars) in terms of the year. Let \( n = 1 \) represent 1990.

75. What did factory sales of pagers total in 1992?

76. When did factory sales of pagers reach $300 million?

77. What was the total of factory sales of pagers for the period 1990–1996?

The Sierpinski triangle is a design using equilateral triangles. The process involves removing smaller triangles from larger triangles by joining the midpoints of the sides of the larger triangles as shown below. Assume that the initial triangle is equilateral with sides 1 unit long.

78. Let \( a_n \) be the number of triangles removed at the \( n \)th stage. Find a rule for \( a_n \). Then find the total number of triangles removed through the 10th stage.

79. Let \( b_n \) be the remaining area of the original triangle at the \( n \)th stage. Find a rule for \( b_n \). Then find the remaining area of the original triangle at the 15th stage.

80. Writing Compare the graphs of \( a_n = 4(2)^n - 1 \) where \( n \) is a positive integer and \( f(x) = 4(2)^x - 1 \) where \( x \) is a real number. Discuss how the graph of a geometric sequence with \( r > 0 \) is similar to and different from the graph of an exponential function.

81. Multi-Step Problem Suppose two computer companies, Company A and Company B, opened in 1991. The revenues of Company A increased arithmetically through 2000, while the revenues of Company B increased geometrically through 2000. In 1996 the revenue of Company A was $523.7 million. In 1996 the revenue of Company B was $65.6 million.

a. The revenues of Company A have a common difference of 55.5. The revenues of Company B have a common ratio of 2. Find a rule for the revenues in the \( n \)th year of each company. Let \( a_1 \) represent 1991.

b. Graph each sequence from part (a).

c. Find the sum of the revenues from 1991 through 2000 for each company.

d. Writing Use a graphing calculator or spreadsheet to find when the revenue of Company B is greater than the revenue of Company A. Write a brief paragraph explaining which company you would rather own. Be sure to refer to your graphs from part (b).

82. Working with Fractions Using the rule for the sum of the first \( n \) terms of a geometric series, write the polynomial as a rational expression.

a. \( 1 + x + x^2 + x^3 + x^4 \)  

b. \( 3x + 6x^3 + 12x^5 + 24x^7 \)
ORDERING NUMBERS  Plot the numbers on a number line. Write the numbers in increasing order. (Review 1.1 for 11.4)

83. \( \frac{3}{2}, \frac{2}{5}, \frac{7}{6}, 1, \frac{6}{7} \)
84. \( \sqrt{5}, 2, -\frac{1}{5}, 0, 3 \)
85. \( -\frac{5}{2}, -1, 1.5, -3.2, -2 \)

SOLVING ALGEBRAICALLY  Solve the inequality algebraically. (Review 5.7)

86. \( x^2 + x - 2 \geq 0 \)
87. \( x^2 - 6x - 7 \leq 0 \)
88. \( x^2 < 36 \)
89. \( -x^2 - 8x < 20 \)
90. \( \frac{1}{2}x^2 + 5x \leq -12 \)

SOLVING EQUATIONS  Solve using any method. Check each solution. (Review 9.6 for 11.4)

92. \( \frac{3}{1 + x} = 8 \)
93. \( \frac{4}{1 - x} = 10 \)
94. \( \frac{-12}{x + 4} = -x \)
95. \( -\frac{24}{x} - x = 11 \)
96. \( \frac{x}{x - 8} = \frac{x}{24} \)
97. \( x + 10 = \frac{x^2}{x - 5} \)

Write the next term in the sequence. Then write a rule for the \( n \)th term. (Lesson 11.1)

1. \( 0, 2, 4, 6, \ldots \)
2. \( 3, 9, 27, 81, \ldots \)
3. \( \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \ldots \)

Find the sum of the series. (Lesson 11.1)

4. \( \sum_{k=0}^{4} k^4 \)
5. \( \sum_{m=1}^{6} (m^2 + 5) \)
6. \( \sum_{n=1}^{5} (n^3 - 1) \)

Write a rule for the \( n \)th term of the arithmetic sequence. Then find \( a_{12} \). (Lesson 11.2)

7. \( 1, 5, 9, 13, \ldots \)
8. \( 34, 25, 16, 7, -2, \ldots \)
9. \( \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \)

10. Find the sum of the first 30 terms of the arithmetic series \( 1.4 + 2.9 + 4.4 + 5.9 + 7.4 + \cdots \). (Lesson 11.2)

Write a rule for the \( n \)th term of the geometric sequence. Then find \( a_{15} \). (Lesson 11.3)

11. \( 2, 10, 50, 250, \ldots \)
12. \( -3, 12, -48, 192, \ldots \)
13. \( 12, 4, \frac{4}{3}, \frac{4}{9}, \ldots \)

14. FAMILY TREE  A portion of John’s parental family tree is shown at the right. Find a rule for the number of people in the \( n \)th generation. If 10 generations of his family have lived in this country, how many people is this? (Lesson 11.3)