3.6 Solving Systems of Linear Equations in Three Variables

What *you should learn*

GOAL 1 Solve systems of linear equations in three variables.

GOAL 2 Use linear systems in three variables to model **real-life** situations, such as a high school swimming meet in **Example 4**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the number of athletes who placed first, second, and third in a track meet in **Ex. 35**.

REAL LIFE

GOAL 1 SOLVING A SYSTEM IN THREE VARIABLES

In Lessons 3.1 and 3.2 you learned how to solve a system of two linear equations in two variables. In this lesson you will learn how to solve a **system of three linear equations** in three variables. Here is an example.

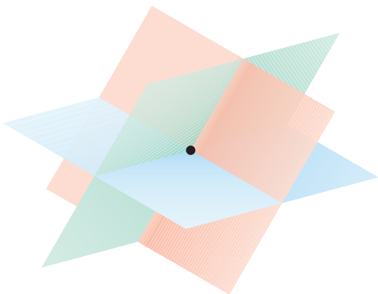
$$x + 2y - 3z = -3 \quad \textbf{Equation 1}$$
$$2x - 5y + 4z = 13 \quad \textbf{Equation 2}$$
$$5x + 4y - z = 5 \quad \textbf{Equation 3}$$

A **solution** of such a system is an ordered triple (x, y, z) that is a solution of all three equations. For instance, $(2, -1, 1)$ is a solution of the system above.

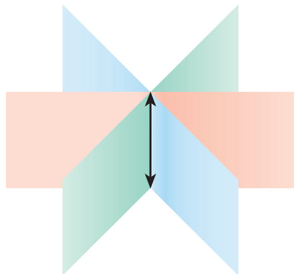
$$2 + 2(-1) - 3(1) = 2 - 2 - 3 = -3 \checkmark$$
$$2(2) - 5(-1) + 4(1) = 4 + 5 + 4 = 13 \checkmark$$
$$5(2) + 4(-1) - 1 = 10 - 4 - 1 = 5 \checkmark$$

From Lesson 3.5 you know that the graph of a linear equation in three variables is a plane. Three planes in space can intersect in different ways.

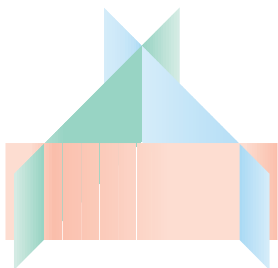
If the planes intersect in a single point, as shown below, the system has exactly one solution.

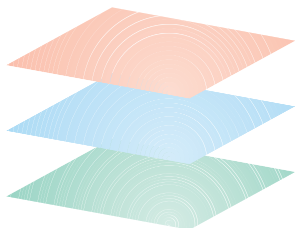


If the planes intersect in a line, as shown below, the system has infinitely many solutions.



If the planes have no point of intersection, the system has no solution. In the example on the left, the planes intersect pairwise, but all three have no points in common. In the example on the right, the planes are parallel.





The linear combination method you learned in Lesson 3.2 can be extended to solve a system of linear equations in three variables.

THE LINEAR COMBINATION METHOD (3-VARIABLE SYSTEMS)

STEP 1 Use the linear combination method to rewrite the linear system in three variables as a linear system in *two* variables.

STEP 2 Solve the new linear system for both of its variables.

STEP 3 Substitute the values found in Step 2 into one of the original equations and solve for the remaining variable.

Note: If you obtain a false equation, such as $0 = 1$, in any of the steps, then the system has no solution. If you do not obtain a false solution, but obtain an identity, such as $0 = 0$, then the system has infinitely many solutions.

EXAMPLE 1 *Using the Linear Combination Method*

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

Solve the system.

$$\begin{aligned} 3x + 2y + 4z &= 11 && \text{Equation 1}\\ 2x - y + 3z &= 4 && \text{Equation 2}\\ 5x - 3y + 5z &= -1 && \text{Equation 3} \end{aligned}$$

SOLUTION

1 Eliminate one of the variables in two of the original equations.

$$\begin{aligned} 3x + 2y + 4z &= 11 && \text{Add 2 times the second}\\ 4x - 2y + 6z &= 8 && \text{equation to the first.}\\ \hline 7x + 10z &= 19 && \text{New Equation 1} \end{aligned}$$

$$\begin{aligned} 5x - 3y + 5z &= -1 && \text{Add } -3 \text{ times the second}\\ -6x + 3y - 9z &= -12 && \text{equation to the third.}\\ \hline -x - 4z &= -13 && \text{New Equation 2} \end{aligned}$$

2 Solve the new system of linear equations in two variables.

$$\begin{aligned} 7x + 10z &= 19 && \text{New Equation 1}\\ -7x - 28z &= -91 && \text{Add 7 times new Equation 2.}\\ \hline -18z &= -72 && \\ z &= 4 && \text{Solve for } z.\\ x &= -3 && \text{Substitute into new Equation 1 or 2 to find } x. \end{aligned}$$

3 Substitute $x = -3$ and $z = 4$ into an original equation and solve for y.

$$\begin{aligned} 2x - y + 3z &= 4 && \text{Equation 2}\\ 2(-3) - y + 3(4) &= 4 && \text{Substitute } -3 \text{ for } x \text{ and 4 for } z.\\ y &= 2 && \text{Solve for } y. \end{aligned}$$

▶ The solution is $x = -3$, $y = 2$, and $z = 4$, or the ordered triple $(-3, 2, 4)$. Check this solution in each of the original equations.

STUDENT HELP

Look Back
For help with solving linear systems with many or no solutions, see p. 150.

EXAMPLE 2 *Solving a System with No Solution*

Solve the system.

$x + y + z = 2$ — **Equation 1**
$3x + 3y + 3z = 14$ — **Equation 2**
$x - 2y + z = 4$ — **Equation 3**

SOLUTION

When you multiply the first equation by -3 and add the result to the second equation, you obtain a false equation.

$$\begin{aligned} -3x - 3y - 3z &= -6 \\ 3x + 3y + 3z &= 14 \\ \hline 0 &= 8 \end{aligned}$$

Add -3 times the first equation to the second.
New Equation 1

▶ Because you obtained a false equation, you can conclude that the original system of equations has no solution.

EXAMPLE 3 *Solving a System with Many Solutions*

Solve the system.

$x + y + z = 2$ — **Equation 1**
$x + y - z = 2$ — **Equation 2**
$2x + 2y + z = 4$ — **Equation 3**

SOLUTION

Rewrite the linear system in three variables as a linear system in two variables.

$$\begin{aligned} x + y + z &= 2 \\ x + y - z &= 2 \\ \hline 2x + 2y &= 4 \end{aligned}$$

Add the first equation to the second.
New Equation 1

$$\begin{aligned} x + y - z &= 2 \\ 2x + 2y + z &= 4 \\ \hline 3x + 3y &= 6 \end{aligned}$$

Add the second equation to the third.
New Equation 2

The result is a system of linear equations in two variables.

$2x + 2y = 4$ — **New Equation 1**
$3x + 3y = 6$ — **New Equation 2**

Solve the new system by adding -3 times the first equation to 2 times the second equation. This produces the identity $0 = 0$. So, the system has infinitely many solutions.

Describe the solution. One way to do this is to divide new Equation 1 by 2 to get $x + y = 2$, or $y = -x + 2$. Substituting this into original Equation 1 produces $z = 0$. So, any ordered triple of the form

$$(x, -x + 2, 0)$$

is a solution of the system. For instance, (0, 2, 0), (1, 1, 0), and (2, 0, 0) are all solutions.

GOAL 2 USING SYSTEMS TO MODEL REAL LIFE

In yesterday's swim meet, **Roosevelt High** dominated in the individual events, with 24 individual-event placers scoring a total of 56 points. A first-place finish scores 5 points, a second-place finish scores 3 points, and a third-place finish scores 1 point. Having as many third-place finishers as first- and second-place finishers combined really shows the team's depth.

EXAMPLE 4 *Writing and Solving a Linear System*

SPORTS Use a system of equations to model the information in the newspaper article. Then solve the system to find how many swimmers finished in each place.

SOLUTION

VERBAL MODEL

	1st-place finishers	+		2nd-place finishers	+		3rd-place finishers	= Total placers
5	1st-place finishers	+	3	2nd-place finishers	+	1	3rd-place finishers	= Total points
				1st-place finishers	+		2nd-place finishers	= 3rd-place finishers

LABELS

1st-place finishers	$= x$	(people)
2nd-place finishers	$= y$	(people)
3rd-place finishers	$= z$	(people)
Total placers	$= 24$	(people)
Total points	$= 56$	(points)

ALGEBRAIC MODEL

$x + y + z = 24$	Equation 1
$5x + 3y + z = 56$	Equation 2
$x + y = z$	Equation 3

Substitute the expression for z from Equation 3 into Equation 1.

$x + y + z = 24$	Write Equation 1.
$x + y + (x + y) = 24$	Substitute $x + y$ for z.
$2x + 2y = 24$	New Equation 1

Substitute the expression for z from Equation 3 into Equation 2.

$5x + 3y + z = 56$	Write Equation 2.
$5x + 3y + (x + y) = 56$	Substitute $x + y$ for z.
$6x + 4y = 56$	New Equation 2

You now have a system of two equations in two variables.

$2x + 2y = 24$	New Equation 1
$6x + 4y = 56$	New Equation 2

▶ When you solve this system you get $x = 4$ and $y = 8$. Substituting these values into original Equation 3 gives you $z = 12$. There were 4 first-place finishers, 8 second-place finishers, and 12 third-place finishers.

GUIDED PRACTICE

Vocabulary Check ✓

1. Give an example of a system of three linear equations in three variables.

Concept Check ✓

2. **ERROR ANALYSIS** A student correctly solves a system of equations in three variables and obtains the equation $0 = 3$. The student concludes that the system has infinitely many solutions. Explain the error in the student's reasoning.

3. Look back at the intersecting planes on page 177. How else can three planes intersect so that the system has infinitely many solutions?

4. Explain how to use the substitution method to solve a system of three linear equations in three variables.

Skill Check ✓

Decide whether the given ordered triple is a solution of the system.

5. (1, 4, 2)
$-2x - y + 5z = 12$
$3x + 2y - z = -7$
$-5x + 4y + 2z = -17$

6. (7, −1, 0)
$-4x + 6y - z = -34$
$-2x - 5y + 8z = -9$
$5x + 2y - 4z = 33$

7. (−2, 3, 3)
$5x - 2y + z = -13$
$x + 4y + 3z = 19$
$-3x + y + 6z = 15$

Use the indicated method to solve the system.

8. linear combination
$x + 5y - z = 16$
$3x - 3y + 2z = 12$
$2x + 4y + z = 20$

9. substitution
$-2x + y + 3z = -8$
$3x + 4y - 2z = 9$
$x + 2y + z = 4$

10. any method
$9x + 5y - z = -11$
$6x + 4y + 2z = 2$
$2x - 2y + 4z = 4$

11. **INVESTMENTS** Your aunt receives an inheritance of \$20,000. She wants to put some of the money into a savings account that earns 2% interest annually and invest the rest in certificates of deposit (CDs) and bonds. A broker tells her that CDs pay 5% interest annually and bonds pay 6% interest annually. She wants to earn \$1000 interest per year, and she wants to put twice as much money in CDs as in bonds. How much should she put in each type of investment?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 944.

LINEAR COMBINATION METHOD **Solve the system using the linear combination method.**

12. $3x + 2y - z = 8$
$-3x + 4y + 5z = -14$
$x - 3y + 4z = -14$

13. $x + 2y + 5z = -1$
$2x - y + z = 2$
$3x + 4y - 4z = 14$

14. $3x + 2y - 3z = -2$
$7x - 2y + 5z = -14$
$2x + 4y + z = 6$

15. $5x - 4y + 4z = 18$
$-x + 3y - 2z = 0$
$4x - 2y + 7z = 3$

16. $x + y - 2z = 5$
$x + 2y + z = 8$
$2x + 3y - z = 13$

17. $-5x + 3y + z = -15$
$10x + 2y + 8z = 18$
$15x + 5y + 7z = 9$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 12–17, 24–33
Examples 2, 3: Exs. 12–33
Example 4: Exs. 18–23, 34–39

SUBSTITUTION METHOD **Solve the system using the substitution method.**

18. $-2x + y + 6z = 1$
$3x + 2y + 5z = 16$
$7x + 3y - 4z = 11$

19. $x - 6y - 2z = -8$
$-x + 5y + 3z = 2$
$3x - 2y - 4z = 18$

20. $x + y + z = 4$
$5x + 5y + 5z = 12$
$x - 4y + z = 9$

21. $x - 3y + 6z = 21$
$3x + 2y - 5z = -30$
$2x - 5y + 2z = -6$

22. $x + y - 2z = 5$
$x + 2y + z = 8$
$2x + 3y - z = 1$

23. $2x - 3y + z = 10$
$y + 2z = 13$
$z = 5$

CHOOSING A METHOD **Solve the system using any algebraic method.**

24. $2x - 2y + z = 3$
$5y - z = -31$
$x + 3y + 2z = -21$

25. $17x - y + 2z = -9$
$x + y - 4z = 8$
$3x - 2y - 12z = 24$

26. $-2x + y + z = -2$
$5x + 3y + 3z = 71$
$4x - 2y - 3z = 1$

27. $x - 9y + 4z = 1$
$-4x + 18y - 8z = -6$
$2x + y - 4z = -3$

28. $2x + y + 2z = 7$
$2x - y + 2z = 1$
$5x + y + 5z = 13$

29. $7x - 3y + 4z = -14$
$8x + 2y - 24z = 18$
$6x - 10y + 8z = -24$

30. $12x + 6y + 7z = -35$
$7x - 5y - 6z = 200$
$x + y = -10$

31. $7x - 10y + 8z = -50$
$-2x - 5y + 12z = -90$
$3x + 4y + 4z = 26$

32. $-2x - 3y - 6z = -26$
$5x + 5y + 4z = 24$
$3x + 4y - 5z = -40$

33. $3x + 3y + z = 30$
$10x - 3y - 7z = 17$
$-6x + 7y + 3z = -49$

34. **FIELD TRIP** You and two friends buy snacks for a field trip. Using the information given in the table, determine the price per pound for mixed nuts, granola, and dried fruit.

Shopper	Mixed nuts	Granola	Dried fruit	Total price
You	1 lb	$\frac{1}{2}$ lb	$\frac{1}{2}$ lb	$5.97
Kenny	$1\frac{1}{3}$ lb	$\frac{1}{4}$ lb	$\frac{3}{2}$ lb	$9.22
Vanessa	$\frac{1}{3}$ lb	$1\frac{1}{2}$ lb	2 lb	$10.96

35. **TRACK MEET** Use a system of linear equations to model the data in the following newspaper article. Solve the system to find how many athletes finished in each place.

> **Lawrence High** prevailed in Saturday's track meet with the help of 20 individual-event placers earning a combined 68 points. A first-place finish earns 5 points, a second-place finish earns 3 points, and a third-place finish earns 1 point. Lawrence had a strong second-place showing, with as many second-place finishers as first- and third-place finishers combined.

36. **CHINESE RESTAURANT** Jeanette, Raj, and Henry go to a Chinese restaurant for lunch and order three different luncheon combination platters. Jeanette orders 2 portions of fried rice and 1 portion of chicken chow mein. Raj orders 1 portion of fried rice, 1 portion of chicken chow mein, and 1 portion of sautéed broccoli. Henry orders 1 portion of sautéed broccoli and 2 portions of chicken chow mein. Jeanette's platter costs $5, Raj's costs $5.25, and Henry's costs $5.75. How much does 1 portion of chicken chow mein cost?

FOCUS ON APPLICATIONS

REAL LIFE — VOTER REGISTRATION
In November of 1996 there were 10.8 million people aged 18–20 years old in the United States. Of these, 5 million people were registered voters and 3.4 million actually voted.

INTERNET — APPLICATION LINK
www.mcdougallittell.com

FURNITURE SALE **In Exercises 37 and 38, use the furniture store ad shown at the right.**

37. Write a system of equations for the three combinations of furniture.

38. What is the price of each piece of furniture?

39. **SOCIAL STUDIES CONNECTION** For several political parties, the table shows the approximate percent of votes for the party's presidential candidate that were cast in 1996 by voters in two regions of the United States. Write and solve a system of equations to find the *total* number of votes for each party (Democrat, Republican, and Other). Use the fact that a total of about 100 million people voted in 1996. ▶ Source: *Statistical Abstract of the United States*

Region	Democrat (%)	Republican (%)	Other parties (%)	Total voters (millions)
Northeast	20	15	20	18
South	30	35	25	31.5

40. **GOING IN REVERSE** Which values should be given to a, b, and c so that the linear system shown has $(-1, 2, -3)$ as its only solution?

$$x + 2y - 3z = a$$
$$-x - y + z = b$$
$$2x + 3y - 2z = c$$

41. **CRITICAL THINKING** Write a system of three linear equations in three variables that has the given number of solutions.

a. one solution **b.** no solution **c.** infinitely many solutions

Test Preparation

42. **MULTI-STEP PROBLEM** You have \$25 to spend on picking 21 pounds of three different types of apples in an orchard. The Empire apples cost \$1.40 per pound, the Red Delicious apples cost \$1.10 per pound, and the Golden Delicious apples cost \$1.30 per pound. You want twice as many Red Delicious apples as the other two kinds combined.

a. Write a system of equations to represent the given information.

b. How many pounds of each type of apple should you buy?

c. *Writing* Create your own situation in which you are buying three different types of fruit. State the total amount of fruit you need, the price of each type of fruit, the amount of money you have to spend, and the desired ratio of one type of fruit to the other two types. Write a system of equations representing your situation. Then solve your system to find the number of pounds of each type of fruit you should buy.

★ Challenge

SYSTEMS OF FOUR EQUATIONS **Solve the system of equations. Describe what you are doing at each step in your solution process.**

43.
$$w + x + y + z = 6$$
$$3w - x + y - z = -3$$
$$2w + 2x - 2y + z = 4$$
$$2w - x - y + z = -4$$

44.
$$2w - x + 5y + z = -3$$
$$3w + 2x + 2y - 6z = -32$$
$$w + 3x + 3y - z = -47$$
$$5w - 2x - 3y + 3z = 49$$

EXTRA CHALLENGE
www.mcdougallittell.com

Mixed Review

Performing an Operation Perform the indicated operation. (Review 1.1 for 4.1)

45. $-10 + 21$

46. $15 - (-1)$

47. $12 \cdot 7$

48. $-2 - (-20)$

49. $-9 + (-7)$

50. $-8(-6)$

51. $-\frac{1}{2} + \frac{4}{5}$

52. $-\frac{1}{3}\left(-\frac{2}{7}\right)$

53. $\frac{3}{4} - 3$

Solving and Graphing Solve the inequality. Then graph your solution. (Review 1.7)

54. $|11 - x| < 20$

55. $|2x + 3| \geq 26$

56. $\left|18 + \frac{1}{2}x\right| \geq 10$

57. $|7 + 8x| > 5$

58. $|5 - x| < 10$

59. $|3x - 1| \leq 30$

60. $|-3x + 6| \geq 12$

61. $|6x + 4| < 40$

62. $|15 - 3x| > 3$

Plotting Points Plot the ordered triple in a three-dimensional coordinate system. (Review 3.5)

63. $(3, 6, 0)$

64. $(-3, -6, -4)$

65. $(-5, 9, 2)$

66. $(-9, 4, -7)$

67. $(6, -2, -6)$

68. $(-8, 5, -6)$

69. $(0, -3, -3)$

70. $(2, 2, -2)$

71. $(-4, -7, -3)$

Quiz 3

Self-Test for Lessons 3.5 and 3.6

Sketch the graph of the equation. Label the points where the graph crosses the *x*-, *y*-, and *z*-axes. (Lesson 3.5)

1. $2x + 5y + 3z = 15$

2. $x + 4y + 16z = 8$

3. $3x + y + z = 10$

4. $3x + 12y + 6z = 9$

5. $5x - 2y + z = 15$

6. $-x + 9y - 3z = 18$

Write the linear equation as a function of *x* and *y*. Then evaluate the function for the given values. (Lesson 3.5)

7. $-x + \frac{1}{2}y + 3z = 18,\ f(2, 0)$

8. $4x + 8y - 8z = -16,\ f(-4, 4)$

9. $20x - 3y - z = 15,\ f(3, -7)$

10. $-2x + y + 6z = 24,\ f(12, 7)$

Solve the system using any algebraic method. (Lesson 3.6)

11. $2x + 4y + 3z = 10$
$3x - y + 6z = 15$
$5x + 2y - z = 25$

12. $3x - 2y + 3z = 11$
$5x + 2y - 2z = 4$
$-x + y + z = -7$

13. $x - 2y + 3z = -9$
$2x + 5y + z = 10$
$3x - 6y + 9z = 12$

14. **State Orchestra** Fifteen band members from your school were selected to play in the state orchestra. Twice as many students who play a wind instrument were selected as students who play a string or percussion instrument. Of the students selected, one fifth play a string instrument. How many students playing each type of instrument were selected to play in the state orchestra? (Lesson 3.6)