2.7 Piecewise Functions

What *you should learn*

GOAL 1 Represent piecewise functions.

GOAL 2 Use piecewise functions to model **real-life** quantities, such as the amount you earn at a summer job in **Example 6**.

Why *you should learn it*

▼ To solve **real-life** problems, such as determining the cost of ordering silk-screen T-shirts in **Exs. 54 and 55**.

REAL LIFE

GOAL 1 REPRESENTING PIECEWISE FUNCTIONS

Up to now in this chapter a function has been represented by a single equation. In many real-life problems, however, functions are represented by a combination of equations, each corresponding to a part of the domain. Such functions are called **piecewise functions**. For example, the piecewise function given by

$$f(x) = \begin{cases} 2x - 1, & \text{if } x \le 1 \\ 3x + 1, & \text{if } x > 1 \end{cases}$$

is defined by two equations. One equation gives the values of $f(x)$ when x is less than or equal to 1, and the other equation gives the values of $f(x)$ when x is greater than 1.

EXAMPLE 1 *Evaluating a Piecewise Function*

Evaluate $f(x)$ when (**a**) $x = 0$, (**b**) $x = 2$, and (**c**) $x = 4$.

$$f(x) = \begin{cases} x + 2, & \text{if } x < 2 \\ 2x + 1, & \text{if } x \ge 2 \end{cases}$$

SOLUTION

a. $f(x) = x + 2$ — Because $0 < 2$, use first equation.

$f(0) = 0 + 2 = 2$ — Substitute 0 for x.

b. $f(x) = 2x + 1$ — Because $2 \ge 2$, use second equation.

$f(2) = 2(2) + 1 = 5$ — Substitute 2 for x.

c. $f(x) = 2x + 1$ — Because $4 \ge 2$, use second equation.

$f(4) = 2(4) + 1 = 9$ — Substitute 4 for x.

EXAMPLE 2 *Graphing a Piecewise Function*

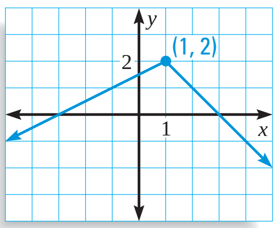
Graph this function: $f(x) = \begin{cases} \frac{1}{2}x + \frac{3}{2}, & \text{if } x < 1 \\ -x + 3, & \text{if } x \ge 1 \end{cases}$

SOLUTION

To the left of $x = 1$, the graph is given by $y = \frac{1}{2}x + \frac{3}{2}$.

To the right of and including $x = 1$, the graph is given by $y = -x + 3$.

The graph is composed of two rays with common initial point $(1, 2)$.

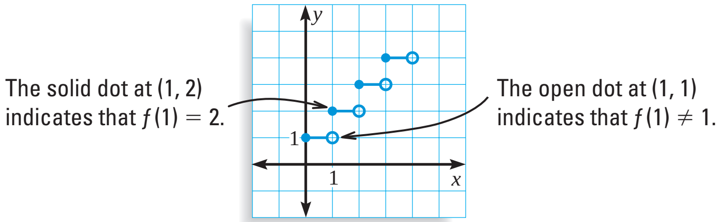


EXAMPLE 3 Graphing a Step Function

Graph this function: $f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 2, & \text{if } 1 \le x < 2 \\ 3, & \text{if } 2 \le x < 3 \\ 4, & \text{if } 3 \le x < 4 \end{cases}$

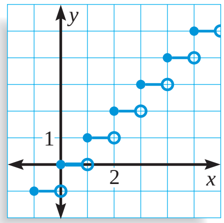
SOLUTION

The graph of the function is composed of four line segments. For instance, the first line segment is given by the equation $y = 1$ and represents the graph when x is greater than or equal to 0 and less than 1.



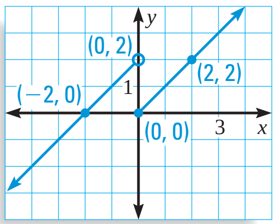
....

The function in Example 3 is called a **step function** because its graph resembles a set of stair steps. Another example of a step function is the *greatest integer function*. This function is denoted by $g(x) = [\![x]\!]$. For every real number x, $g(x)$ is the greatest integer less than or equal to x. The graph of $g(x)$ is shown at the right. Note that in Example 3 the function f could have been written as $f(x) = [\![x]\!] + 1, 0 \le x < 4$.



EXAMPLE 4 Writing a Piecewise Function

Write equations for the piecewise function whose graph is shown.



SOLUTION

To the left of $x = 0$, the graph is part of the line passing through $(-2, 0)$ and $(0, 2)$. An equation of this line is given by:

$$y = x + 2$$

To the right of and including $x = 0$, the graph is part of the line passing through $(0, 0)$ and $(2, 2)$. An equation of this line is given by:

$$y = x$$

▶ The equations for the piecewise function are:

$$f(x) = \begin{cases} x + 2, & \text{if } x < 0 \\ x, & \text{if } x \ge 0 \end{cases}$$

Note that $f(x) = x + 2$ does *not* correspond to $x = 0$ because there is an *open* dot at $(0, 2)$, but $f(x) = x$ *does* correspond to $x = 0$ because there is a *solid* dot at $(0, 0)$.

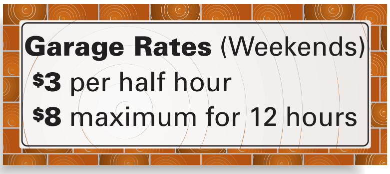
GOAL 2 USING PIECEWISE FUNCTIONS IN REAL LIFE

REAL LIFE
Urban Parking

EXAMPLE 5 *Using a Step Function*

a. Write and graph a piecewise function for the parking charges shown on the sign.

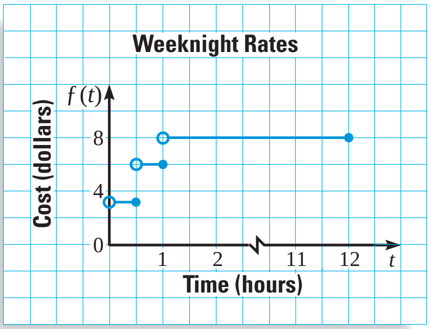
b. What are the domain and range of the function?



SOLUTION

a. For times up to one half hour, the charge is \$3. For each additional half hour (or portion of a half hour), the charge is an additional \$3 until you reach \$8. Let t represent the number of hours you park. The piecewise function and graph are:

$$f(t) = \begin{cases} 3, & \text{if } 0 < t \le 0.5 \\ 6, & \text{if } 0.5 < t \le 1 \\ 8, & \text{if } 1 < t \le 12 \end{cases}$$



b. The domain is $0 < t \le 12$, and the range consists of 3, 6, 8

REAL LIFE
Wages

EXAMPLE 6 *Using a Piecewise Function*

You have a summer job that pays time and a half for overtime. That is, if you work more than 40 hours per week, your hourly wage for the extra hours is 1.5 times your normal hourly wage of \$7.

a. Write and graph a piecewise function that gives your weekly pay P in terms of the number h of hours you work.

b. How much will you get paid if you work 45 hours?

SOLUTION

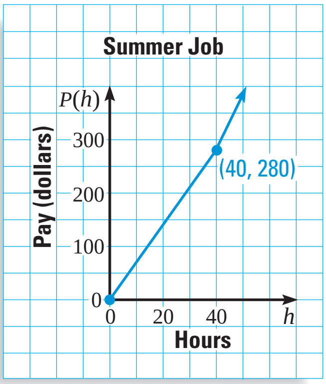
a. For up to 40 hours your pay is given by $7h$.
For over 40 hours your pay is given by:

$$7(40) + 1.5(7)(h - 40) = 10.5h - 140$$

▶ The piecewise function is:

$$P(h) = \begin{cases} 7h, & \text{if } 0 \le h \le 40 \\ 10.5h - 140, & \text{if } h > 40 \end{cases}$$

The graph of the function is shown. Note that for up to 40 hours the rate of change is \$7 per hour, but for over 40 hours the rate of change is \$10.50 per hour.



b. To find how much you will get paid for working 45 hours, use the equation $P(h) = 10.5h - 140$.

$$P(45) = 10.5(45) - 140 = 332.5$$

▶ You will earn \$332.50.

GUIDED PRACTICE

Vocabulary Check ✓

1. Define piecewise function and step function. Give an example of each.

Concept Check ✓

2. Look back at Example 3. What does a solid dot on the graph of a step function indicate? What does an open dot indicate?

Tell whether the statement is *True* or *False*. Explain.

3. In the graph of a piecewise function, the separate pieces are always connected.

4. $f(x) = \begin{cases} 2, & \text{if } 1 \le x < 2 \\ 4, & \text{if } 2 \le x < 3 \\ 6, & \text{if } 3 \le x < 4 \end{cases}$ can be rewritten as $f(x) = 2[\![x]\!],\ 1 \le x < 4$.

Skill Check ✓

Evaluate $f(x) = \begin{cases} 3x - 1, & \text{if } x \le 4 \\ 2x + 7, & \text{if } x > 4 \end{cases}$ for the given value of x.

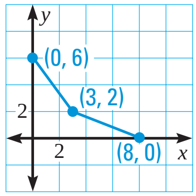
5. $x = 10$ **6.** $x = -\frac{1}{3}$ **7.** $x = 4$ **8.** $x = -2$

Graph the function.

9. $f(x) = \begin{cases} 2x + 1, & \text{if } x < 1 \\ -x + 4, & \text{if } x \ge 1 \end{cases}$

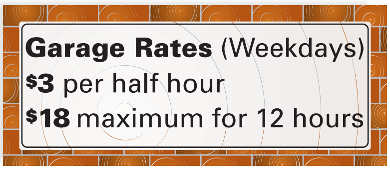
10. $f(x) = \begin{cases} 4, & \text{if } 0 \le x < 2 \\ 5, & \text{if } 2 \le x < 4 \\ 6, & \text{if } 4 \le x < 6 \end{cases}$

11. Write equations for the piecewise function whose graph is shown.



Ex. 11

12. **PARKING RATES** The weekday parking rates for a garage are shown. Write and graph a piecewise function for the weekday parking charges at that garage.



PRACTICE AND APPLICATIONS

STUDENT HELP
Extra Practice to help you master skills is on p. 942.

EVALUATING FUNCTIONS **Evaluate the function for the given value of x.**

$f(x) = \begin{cases} 5x - 1, & \text{if } x < -2 \\ x - 9, & \text{if } x \ge -2 \end{cases}$

13. $f(-4)$ **14.** $f(-2)$
15. $f(0)$ **16.** $f(5)$

$h(x) = \begin{cases} \frac{1}{2}x - 10, & \text{if } x \le 6 \\ -x - 1, & \text{if } x > 6 \end{cases}$

17. $h(1)$ **18.** $h(-10)$
19. $h(6)$ **20.** $h(0)$

STUDENT HELP
HOMEWORK HELP
Example 1: Exs. 13–20
Example 2: Exs. 21–26
Example 3: Exs. 27–32
Example 4: Exs. 35–40
Examples 5 and 6: Exs. 50–59

GRAPHING FUNCTIONS **Graph the function.**

21. $f(x) = \begin{cases} 2x, & \text{if } x \ge 1 \\ -x + 3, & \text{if } x < 1 \end{cases}$

22. $f(x) = \begin{cases} x + 6, & \text{if } x \le -3 \\ -\frac{2}{3}x - 3, & \text{if } x > -3 \end{cases}$

23. $f(x) = \begin{cases} 2x + 13, & \text{if } x \ge -5 \\ x + \frac{1}{2}, & \text{if } x < -5 \end{cases}$

24. $f(x) = \begin{cases} -x, & \text{if } x > 2 \\ x - 4, & \text{if } x \le 2 \end{cases}$

25. $f(x) = \begin{cases} 3x - 14, & \text{if } x \le 4 \\ -2x + 6, & \text{if } x > 4 \end{cases}$

26. $f(x) = \begin{cases} x - 8, & \text{if } x < 9 \\ \frac{1}{3}x - 2, & \text{if } x \ge 9 \end{cases}$

GRAPHING STEP FUNCTIONS **Graph the step function.**

27. $f(x) = \begin{cases} 3, & \text{if } -1 \le x < 2 \\ 5, & \text{if } 2 \le x < 4 \\ 8, & \text{if } 4 \le x < 9 \\ 10, & \text{if } 9 \le x < 12 \end{cases}$

28. $f(x) = \begin{cases} 6.5, & \text{if } -4 \le x < -2 \\ 4.1, & \text{if } -2 \le x < 1 \\ 0.9, & \text{if } 1 \le x < 3 \\ -2.1, & \text{if } 3 \le x < 6 \end{cases}$

29. $f(x) = \begin{cases} -1, & \text{if } 0 \le x < 1 \\ -3, & \text{if } 1 \le x < 2 \\ -5, & \text{if } 2 \le x < 3 \\ -7, & \text{if } 3 \le x < 4 \\ -9, & \text{if } 4 \le x < 5 \end{cases}$

30. $f(x) = \begin{cases} 4, & \text{if } -10 < x \le -8 \\ 6, & \text{if } -8 < x \le -6 \\ 8, & \text{if } -6 < x \le -4 \\ 9.1, & \text{if } -4 < x \le -2 \\ 10, & \text{if } -2 < x \le 0 \end{cases}$

SPECIAL STEP FUNCTIONS **Graph the special step function. Then explain how you think the function got its name.**

31. CEILING FUNCTION

$f(x) = \lceil x \rceil = \begin{cases} \dots \\ 1, & \text{if } 0 < x \le 1 \\ 2, & \text{if } 1 < x \le 2 \\ 3, & \text{if } 2 < x \le 3 \\ \dots \end{cases}$

32. ROUNDING FUNCTION

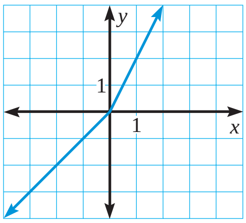
$f(x) = \text{ROUND}(x) = \begin{cases} \dots \\ 1, & \text{if } 0.5 \le x < 1.5 \\ 2, & \text{if } 1.5 \le x < 2.5 \\ 3, & \text{if } 2.5 \le x < 3.5 \\ \dots \end{cases}$

33. CRITICAL THINKING Look back at Example 2. How would the graph of the function change if $<$ was replaced with $\le$ and $\ge$ was replaced with $>$? Explain your answer.

34. CRITICAL THINKING Look back at Example 3. How would the graph of the function change if each $\le$ was replaced with $<$ and each $<$ was replaced with $\le$? Explain your answer.

WRITING PIECEWISE FUNCTIONS **Write equations for the piecewise function whose graph is shown.**

35.



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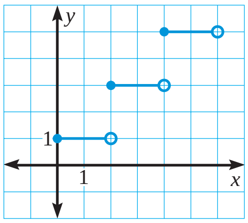

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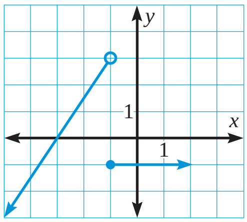

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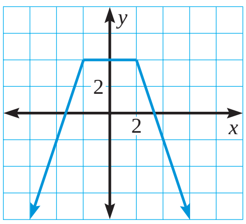

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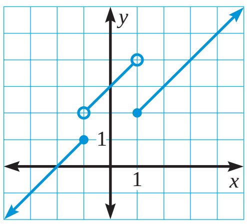

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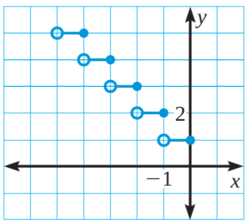


STUDENT HELP

KEYSTROKE HELP Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

GREATEST INTEGER FUNCTION **On many graphing calculators $\llbracket x \rrbracket$ is denoted by int(x). Use a graphing calculator to graph the function.**

41. $g(x) = \llbracket x \rrbracket$

42. $g(x) = \llbracket 2x \rrbracket$

43. $g(x) = \llbracket x \rrbracket - 1$

44. $g(x) = \llbracket x + 3 \rrbracket$

45. $g(x) = 6\llbracket x \rrbracket$

46. $g(x) = \llbracket 3x \rrbracket + 4$

47. $g(x) = 4\llbracket x + 7 \rrbracket$

48. $g(x) = -\llbracket x \rrbracket$

49. $g(x) = 3\llbracket x - 2 \rrbracket + 5$

POSTAL RATES **In Exercises 50 and 51, use the following information.**
As of January 10, 1999, the cost C (in dollars) of sending next-day mail using the United States Postal Service, depending on the weight x (in ounces) of a package up to five pounds, is given by the function below.

INTERNET **DATA UPDATE** of United States Postal Service data at www.mcdougallittell.com

$$C(x) = \begin{cases} 11.75, & \text{if } 0 < x \le 8 \\ 15.75, & \text{if } 8 < x \le 32 \\ 18.50, & \text{if } 32 < x \le 48 \\ 21.25, & \text{if } 48 < x \le 64 \\ 24.00, & \text{if } 64 < x \le 80 \end{cases}$$

50. Graph the function.

51. Identify the domain and range of the function.

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 52 and 53.

PHOTOCOPY RATES **In Exercises 52 and 53, use the function given for the cost C (in dollars) of making x photocopies at a copy shop.**

$$C(x) = \begin{cases} 0.15x, & \text{if } 0 < x \le 25 \\ 0.10x, & \text{if } 26 \le x \le 100 \\ 0.07x, & \text{if } 101 \le x \le 500 \\ 0.05x, & \text{if } 501 \le x \end{cases}$$

52. Graph the function.

53. **VISUAL THINKING** Use your graph to explain why it would not be cost-effective to make 450 photocopies.

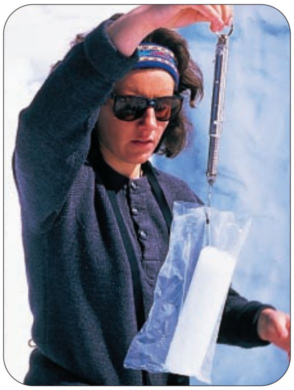
SILK-SCREEN T-SHIRTS **In Exercises 54 and 55, use the following silk-screen shop charges.**

- An initial charge of \$20 to create the silk screen
- \$17.00 per shirt for orders of 50 or fewer shirts
- \$15.80 per shirt for orders of more than 50 shirts

54. Write a piecewise function that gives the cost C for an order of x shirts.

55. Graph the function.

FOCUS ON APPLICATIONS



REAL LIFE **SNOWSTORM**
By weighing snow at the end of a snowstorm you can determine the water content of the snow. This information is one of the factors used to determine avalanche warnings.

INTERNET **APPLICATION LINK**
www.mcdougallittell.com

SOCIAL SECURITY **In Exercises 56 and 57, use the following information.**
The amount of Social Security tax you pay, part of your Federal Insurance Contributions Act (FICA) deductions, depends on your annual income. As of 1999 you pay 6.2% of your income if it is less than \$72,600. If your income is at least \$72,600, you pay a fixed amount of \$4501.20.

INTERNET **DATA UPDATE** of Social Security Administration data at www.mcdougallittell.com

56. Write and graph a piecewise function that gives the Social Security tax.

57. How much Social Security tax do you pay if you make \$30,000 per year?

SNOWSTORM **In Exercises 58 and 59, use the following information.**
During a nine hour snowstorm it snows at a rate of 1 inch per hour for the first two hours, at a of rate of 2 inches per hour for the next six hours, and at a rate of 1 inch per hour for the final hour.

58. Write and graph a piecewise function that gives the depth of the snow during the snowstorm.

59. How many inches of snow accumulated from the storm?

Test Preparation

QUANTITATIVE COMPARISON In Exercises 60 and 61, choose the statement that is true about the given quantities.

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
60.	$f(3)$ where $f(x) = \begin{cases} 2x - 7, & \text{if } x \le 1 \\ -x + 9, & \text{if } x > 1 \end{cases}$	$f(2)$ where $f(x) = \begin{cases} x + 2, & \text{if } x < 8 \\ 3x - 3, & \text{if } x \ge 8 \end{cases}$
61.	$f(0)$ where $f(x) = \begin{cases} 5x + 1, & \text{if } x < 9 \\ 6x - 4, & \text{if } x \ge 9 \end{cases}$	$f(-4)$ where $f(x) = \begin{cases} 9, & \text{if } x \le -4 \\ 11, & \text{if } x > -4 \end{cases}$

★ Challenge

62. **SCUBA DIVING** The time t (in minutes) that a person may safely scuba dive without having to decompress while surfacing is determined by the depth d (in feet) of the dive. Using the information below, write and graph a piecewise inequality that describes the time limits for scuba divers at various depths.

- For depths from 40 feet (the minimum depth requiring decompression) to $53\frac{1}{3}$ feet, the time must not exceed 600 minutes minus ten times the depth.
- For depths greater than $53\frac{1}{3}$ feet to less than 90 feet, the time must not exceed 120 minutes minus the depth.
- For depths from 90 feet to 130 feet (the maximum safe depth for a recreational diver), the time must not exceed 75 minutes minus one half the depth.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

SOLVING EQUATIONS Solve the equation. (Review 1.7 for 2.8)

63. $|9 + 4x| = 15$ **64.** $|7x + 3| = 11$ **65.** $|21 - 2x| = 9$

66. $|2x + 8| = 1$ **67.** $\left|\frac{1}{2}x - 5\right| = 11$ **68.** $\left|1 - \frac{3}{4}x\right| = 6$

SCATTER PLOTS Draw a scatter plot of the data. Then tell whether the data have a *positive*, a *negative*, or *relatively no correlation*. (Review 2.5)

69.

x	−8	−8	−7	−6	−5	−4	−4	−2	−2	−1
y	−2	−8	−5	−7	−1	−4	−8	−1	−3	−7

70.

x	1	1.5	1.5	2.5	3	3.5	5	5.5	7	8
y	9	8	6	5	6	4	2	3	1	2

71. **SLEEPING BAGS** To be comfortable, sleeping bags rated for −40°F have 3.5 inches of insulation, and those rated for 40°F have 1.5 inches. Write a linear model for the amount a of insulation needed to be comfortable at temperature T. How much insulation would you need to be comfortable at 0°F? (Review 2.4)