Solving Trigonometric Equations

GOAL 1 SOLVING A TRIGONOMETRIC EQUATION

In Lesson 14.3 you verified trigonometric identities. In this lesson you will solve trigonometric equations. To see the difference, consider the following equations:

\[
\sin^2 x + \cos^2 x = 1 \quad \text{Equation 1}
\]

\[
\sin x = 1 \quad \text{Equation 2}
\]

Equation 1 is an identity because it is true for all real values of \( x \). Equation 2, however, is true only for some values of \( x \). When you find these values, you are solving the equation.

EXAMPLE 1 Solving a Trigonometric Equation

Solve \( 2 \sin x - 1 = 0 \).

**Solution**

First isolate \( \sin x \) on one side of the equation.

\[
2 \sin x - 1 = 0 \quad \text{Write original equation.}
\]

\[
2 \sin x = 1 \quad \text{Add 1 to each side.}
\]

\[
\sin x = \frac{1}{2} \quad \text{Divide each side by 2.}
\]

One solution of \( \sin x = \frac{1}{2} \) in the interval \( 0 \leq x < 2\pi \) is \( x = \sin^{-1} \frac{1}{2} = \frac{\pi}{6} \). Another such solution is \( x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \).

Moreover, because \( y = \sin x \) is a periodic function, there are infinitely many other solutions. You can write the general solution as

\[
x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi
\]

where \( n \) is any integer.

**Check** You can check your answer graphically. Graph \( y = \sin x \) and \( y = \frac{1}{2} \) in the same coordinate plane and find the points where the graphs intersect.

You can see that there are infinitely many such points.
Chapter 14  Trigonometric Graphs, Identities, and Equations

**Example 2  Solving a Trigonometric Equation in an Interval**

Solve \(4 \tan^2 x - 1 = 0\) in the interval \(0 \leq x < 2\pi\).

**Solution**

\[
4 \tan^2 x - 1 = 0 \\
4 \tan^2 x = 1 \\
\tan^2 x = \frac{1}{4} \\
\tan x = \pm \frac{1}{2}
\]

Use a calculator to find values of \(x\) for which \(\tan x = \pm \frac{1}{2}\), as shown at the right.

The general solution of the equation is

\[x \approx 0.464 + n\pi\]

or

\[x \approx -0.464 + n\pi\]

where \(n\) is any integer. The solutions that are in the interval \(0 \leq x < 2\pi\) are:

\[x \approx 0.464\]

\[x \approx 0.464 + \pi \approx 3.61\]

\[x \approx -0.464 + \pi \approx 2.628\]

\[x \approx -0.464 + 2\pi \approx 5.82\]

**Check** Check these solutions by substituting them back into the original equation.

**Example 3  Factoring to Solve a Trigonometric Equation**

Solve \(\sin^2 x \cos x = 4 \cos x\).

**Solution**

\[
\sin^2 x \cos x = 4 \cos x \\
\sin^2 x \cos x - 4 \cos x = 0 \\
\cos x (\sin^2 x - 4) = 0 \\
\cos x (\sin x + 2)(\sin x - 2) = 0
\]

Set each factor equal to 0 and solve for \(x\), if possible.

\[\cos x = 0 \quad \sin x + 2 = 0 \quad \sin x - 2 = 0\]

\[x = \frac{\pi}{2} \quad \text{or} \quad x = \frac{3\pi}{2} \quad \sin x = -2 \quad \sin x = 2\]

Because neither \(\sin x = -2\) nor \(\sin x = 2\) has a solution, the only solutions in the interval \(0 \leq x < 2\pi\) are \(x = \frac{\pi}{2}\) and \(x = \frac{3\pi}{2}\).

The general solution is \(x = \frac{\pi}{2} + 2n\pi\) or \(x = \frac{3\pi}{2} + 2n\pi\) where \(n\) is any integer.
EXAMPLE 4 Using the Quadratic Formula

Solve \( \cos^2 x - 4 \cos x + 1 = 0 \) in the interval \( 0 \leq x \leq \pi \).

**Solution**

Since the equation is in the form \( au^2 + bu + c = 0 \), you can use the quadratic formula to solve for \( u = \cos x \).

\[ \cos^2 x - 4 \cos x + 1 = 0 \]

Write original equation.

\[ \cos x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \]

Quadratic formula

\[ \cos x = \frac{4 \pm \sqrt{16 - 4}}{2} \]

\[ \cos x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} \]

Simplify.

\[ \approx 3.73 \text{ or } 0.268 \]

Use a calculator.

\[ x \approx \cos^{-1} 3.73 \text{ or } x \approx \cos^{-1} 0.268 \]

Use inverse cosine.

\[ x \approx 1.30 \]

Use a calculator if possible.

In the interval \( 0 \leq x \leq \pi \), the only solution is \( x \approx 1.30 \). Check this in the original equation.

When solving a trigonometric equation, it is possible to obtain extraneous solutions. Therefore, you should always check your solutions in the original equation.

EXAMPLE 5 An Equation with Extraneous Solutions

Solve \( 1 - \cos x = \sqrt{3} \sin x \) in the interval \( 0 \leq x < 2\pi \).

**Solution**

\[ 1 - \cos x = \sqrt{3} \sin x \]

Write original equation.

\[ (1 - \cos x)^2 = (\sqrt{3} \sin x)^2 \]

Square both sides.

\[ 1 - 2 \cos x + \cos^2 x = 3 \sin^2 x \]

Multiply.

\[ 1 - 2 \cos x + \cos^2 x = 3(1 - \cos^2 x) \]

Pythagorean identity

\[ 4 \cos^2 x - 2 \cos x - 2 = 0 \]

Quadratic form

\[ 2 \cos^2 x - \cos x - 1 = 0 \]

Factor.

\[ (2 \cos x + 1)(\cos x - 1) = 0 \]

Zero product property

\[ 2 \cos x + 1 = 0 \text{ or } \cos x - 1 = 0 \]

Solve for \( \cos x \).

\[ \cos x = -\frac{1}{2} \text{ or } \cos x = 1 \]

\[ x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3} \]

Solve for \( x \).

\[ x = 0 \]

The apparent solution \( x = \frac{4\pi}{3} \) does not check in the original equation. The only solutions in the interval \( 0 \leq x < 2\pi \) are \( x = 0 \) and \( x = \frac{2\pi}{3} \).
SOLVING TRIGONOMETRIC EQUATIONS IN REAL LIFE

**EXAMPLE 6** Solving a Real-Life Trigonometric Equation

**METEOROLOGY** The number $h$ of hours of sunlight per day in Prescott, Arizona, can be modeled by

$$h = 2.325 \sin \left( \frac{\pi}{6} (t - 2.667) \right) + 12.155$$

where $t$ is measured in months and $t = 0$ represents January 1. On which days of the year are there 13 hours of sunlight in Prescott?  

**Solution**

**Method 1** Substitute 13 for $h$ in the model and solve for $t$.

$$2.325 \sin \left( \frac{\pi}{6} (t - 2.667) \right) + 12.155 = 13$$

$$2.325 \sin \left( \frac{\pi}{6} (t - 2.667) \right) = 0.845$$

$$\sin \left( \frac{\pi}{6} (t - 2.667) \right) = 0.363$$

$$\frac{\pi}{6} (t - 2.667) = \arcsin(0.363)$$

$$\frac{\pi}{6} (t - 2.667) = 0.371$$

$$t - 2.667 = 0.709$$

$$t = 3.38$$

$$t = 7.96$$

The time $t = 3.38$ represents 3 full months plus $0.38(30) = 11$ days, or April 11. Likewise, the time $t = 7.96$ represents 7 full months plus $0.96(31) = 30$ days, or August 30. (Notice that these two days occur about 70 days before and after June 21, which is the date of the summer solstice, the longest day of the year.)

**Method 2** Use a graphing calculator. Graph the equations

$$y = 2.325 \sin \left( \frac{\pi}{6} (x - 2.667) \right) + 12.155$$

$$y = 13$$

in the same viewing window. Then use the *Intersect* feature to find the points of intersection.

From the screens above, you can see that $t = 3.38$ or $t = 7.96$. The time $t = 3.38$ is about April 11, and the time $t = 7.96$ is about August 30.
1. What is the difference between a trigonometric equation and a trigonometric identity?

2. Name several techniques for solving trigonometric equations.

3. **ERROR ANALYSIS** Describe the error(s) in the calculations shown.

   \[ \cos^2 x = \frac{1}{2} \cos x \]
   \[ \cos x = \frac{1}{2} \]
   \[ x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3} \text{ Solution for } 0 \leq x < 2\pi \]
   \[ x = \frac{\pi}{3} + 2\pi \text{ or } x = \frac{5\pi}{3} + 2\pi \text{ General solution} \]

4. Solve the equation in the interval \(0 \leq x < 2\pi\).

   4. \(2 \cos x + 4 = 5\)
   5. \(3 \sec^2 x - 4 = 0\)
   6. \(\tan^2 x = \cos x \tan^2 x\)
   7. \(5 \cos x - \sqrt{3} = 3 \cos x\)

**Find the general solution of the equation.**

   8. \(3 \csc x + 5 = 0\)
   9. \(4 \sin x = \sqrt{3}\)
   10. \(1 + \tan^2 x = 6 - 2 \sec^2 x\)
   11. \(2 - 2 \cos^2 x = 5 \sin x + 3\)

**METEOROLOGY** Look back at the model in Example 6 on page 858. On which days of the year are there 10 hours of sunlight in Prescott, Arizona?

**PRACTICE AND APPLICATIONS**

**CHECKING SOLUTIONS** Verify that the given \(x\)-value is a solution of the equation.

   13. \(5 + 4 \cos x - 1 = 0, x = \pi\)
   14. \(\csc x - 2 = 0, x = \frac{5\pi}{6}\)
   15. \(4 \cos^2 x - 3 = 0, x = \frac{\pi}{4}\)
   16. \(3 \tan^3 x - 3 = 0, x = \frac{\pi}{6}\)
   17. \(2 \sin^4 x - \sin^2 x = 0, x = \frac{5\pi}{4}\)
   18. \(2 \cot^4 x - \cot^2 x - 15 = 0, x = \frac{13\pi}{6}\)

**SOLVING** Find the general solution of the equation.

   19. \(2 \cos x - 1 = 0\)
   20. \(3 \tan x - \sqrt{3} = 0\)
   21. \(\sin x = \sin (-x) + 1\)
   22. \(4 \cos x = 2 \cos x + 1\)
   23. \(4 \sin^2 x - 2 = 0\)
   24. \(9 \tan^2 x - 3 = 0\)
   25. \(\sin x \cos x - 2 \cos x = 0\)
   26. \(\sqrt{2} \cos x \sin x - \cos x = 0\)
   27. \(2 \sin^2 x - \sin x = 1\)
   28. \(0 = \cos^2 x - 5 \cos x + 1\)
   29. \(1 - \sin x = \sqrt{3} \cos x\)
   30. \(\sqrt{3} \sin x = 2 \sin x - 1\)
   31. \(\cos x - 1 = -\cos x\)
   32. \(6 \sin x = \sin x + 3\)

**GUIDED PRACTICE**

**Vocabulary Check**

1. What is the difference between a trigonometric equation and a trigonometric identity?

**Concept Check**

2. Name several techniques for solving trigonometric equations.

**ERROR ANALYSIS** Describe the error(s) in the calculations shown.

**Skill Check**

**Find the general solution of the equation.**

4. \(2 \cos x + 4 = 5\)
5. \(3 \sec^2 x - 4 = 0\)
6. \(\tan^2 x = \cos x \tan^2 x\)
7. \(5 \cos x - \sqrt{3} = 3 \cos x\)

**Find the general solution of the equation.**

8. \(3 \csc x + 5 = 0\)
9. \(4 \sin x = \sqrt{3}\)
10. \(1 + \tan^2 x = 6 - 2 \sec^2 x\)
11. \(2 - 2 \cos^2 x = 5 \sin x + 3\)

**METEOROLOGY** Look back at the model in Example 6 on page 858. On which days of the year are there 10 hours of sunlight in Prescott, Arizona?
**SOLVING** Solve the equation in the interval $0 \leq x < 2\pi$. Check your solutions.

33. $5 \cos x - 3 = 0$  
34. $3 \sin x = \sin x - 1$  
35. $\tan^2 x - 3 = 0$  
36. $10 \tan x - 5 = 0$  
37. $2 \cos^2 x - \sin x - 1 = 0$  
38. $\cos^3 x = \cos x$  
39. $\sec^2 x - 2 = 0$  
40. $\tan^2 x = \sin x \sec x$  
41. $2 \cos x = \sec x$  
42. $\cos x \csc^2 x + 3 \cos x = 7 \cos x$

**APPROXIMATING SOLUTIONS** Use a graphing calculator to approximate the solutions of the equation in the interval $0 \leq x < 2\pi$.

43. $3 \tan x + 1 = 13$  
44. $8 \cos x + 3 = 4$  
45. $4 \sin x = -2 \sin x - 5$  
46. $3 \sin x + 5 \cos x = 4$

**FINDING INTERCEPTS** Find the $x$-intercepts of the graph of the given function in the interval $0 \leq x < 2\pi$.

47. $y = 2 \sin x + 1$  
48. $y = 2 \tan^2 x - 6$  
49. $y = \sec^2 x - 1$  
50. $y = -3 \cos x + \sin x$

**FINDING INTERSECTION POINTS** Find the points of intersection of the graphs of the given functions in the interval $0 \leq x < 2\pi$.

51. $y = \sqrt{3} \tan^2 x$  
52. $y = 9 \cos^2 x$  
53. $y = \tan x \sin x$  
54. $y = \sin^2 x$  
55. $y = 2 - \sin x \tan x$  
56. $y = 4 \cos^2 x$  
57. $y = 2 \sin x - 1$  
58. $y = \cos x$  
59. $y = 4 \cos x - 1$

57. **Ocean Tides** The tide, or depth of the ocean near the shore, changes throughout the day. The depth of the Bay of Fundy can be modeled by

$$d = 35 - 28 \cos \frac{\pi}{6.2} t$$

where $d$ is the water depth in feet and $t$ is the time in hours. Consider a day in which $t = 0$ represents 12:00 A.M. For that day, when do the high and low tides occur? At what time(s) is the water depth 3 feet?

58. **Position of the Sun** Cheyenne, Wyoming, has a latitude of 41°N. At this latitude, the position of the sun at sunrise can be modeled by

$$D = 31 \sin \left( \frac{2\pi}{365} t - 1.4 \right)$$

where $t$ is the time in days and $t = 1$ represents January 1. In this model, $D$ represents the number of degrees north or south of due east that the sun rises. Use a graphing calculator to determine the days that the sun is more than 20° north of due east at sunrise.

59. **Meteorology** A model for the average daily temperature $T$ (in degrees Fahrenheit) in Kansas City, Missouri, is given by

$$T = 54 + 25.2 \sin \left( \frac{2\pi}{12} t + 4.3 \right)$$

where $t$ is measured in months and $t = 0$ represents January 1. What months have average daily temperatures higher than 70°F? Do any months have average daily temperatures below 20°F? Source: National Climatic Data Center
60. **MULTIPLE CHOICE** What is the general solution of the equation \(\cos x + \sqrt{2} = -\cos x\)? Assume \(n\) is an integer.

A. \(x = \frac{\pi}{4} + 2n\pi\)
B. \(x = \frac{3\pi}{4} + 2n\pi\) or \(x = \frac{7\pi}{4} + 2n\pi\)
C. \(x = \frac{3\pi}{4} + 2n\pi\)
D. \(x = \frac{3\pi}{4} + 2n\pi\) or \(x = \frac{5\pi}{4} + 2n\pi\)
E. \(x = \frac{3\pi}{4} + n\pi\) or \(x = \frac{5\pi}{4} + n\pi\)

61. **MULTIPLE CHOICE** Find the points of intersection of the graphs of \(y = 2 + \sin x\) and \(y = 3 - \sin x\) in the interval \(0 \leq x < 2\pi\).

A. \(\left(\frac{\pi}{6}, \frac{5\pi}{2}\right), \left(\frac{5\pi}{6}, \frac{5\pi}{2}\right)\)
B. \(\left(\frac{\pi}{3}, \frac{5\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{3}, \frac{4\pi}{3}, \frac{5}{2}\right)\)
C. \(\left(\frac{\pi}{3}, \frac{5\pi}{6}, \frac{1}{2}\right)\)
D. \(\left(\frac{\pi}{3}, \frac{4\pi}{6}, \frac{1}{2}\right)\)
E. \(\left(\frac{\pi}{6}, \frac{5\pi}{6}, \frac{1}{2}\right)\)

**MATRICES** In Exercises 62 and 63, use the following information.

Matrix multiplication can be used to rotate a point \((x, y)\) counter clockwise about the origin through an angle \(\theta\). The coordinates of the resulting point \((x', y')\) are determined by the following matrix equation:

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

62. The point \((4, 1)\) is rotated counter clockwise about the origin through an angle of \(\frac{\pi}{3}\). What are the coordinates of the resulting point?

63. Through what angle \(\theta\) must the point \((2, 4)\) be rotated to produce \((x', y') = (-2 + \sqrt{3}, 1 + 2\sqrt{3})\)?

**EXTRA CHALLENGE**

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**MIXED REVIEW**

**USING COMPLEMENTS** Two six-sided dice are rolled. Find the probability of the given event. (Review 12.4)

64. The sum is less than or equal to 10.  
65. The sum is not 4.  
66. The sum is not 2 or 12.  
67. The sum is greater than 3.

**GRAPHING** Graph the function. (Review 14.1, 14.2 for 14.5)

68. \(y = \sin 3x\)  
69. \(y = 2\cos 4x\)  
70. \(y = 10\sin 2x\)  
71. \(y = 3\cos \frac{1}{2}x\)  
72. \(y = \frac{1}{4}\tan x\)  
73. \(y = 3\tan \frac{1}{2}x - 2\)  
74. \(y = \cos \frac{1}{2}x + \pi\)  
75. \(y = 3 + \tan \left(x - \frac{3\pi}{2}\right)\)  
76. \(y = -\sin 3\pi(x + 4) + 1\)

**SURVEYING** Suppose you are trying to determine the width \(w\) of a small pond. You stand at a point 43 feet from one end of the pond and 50 feet from the other end. The angle formed by your lines of sight to each end of the pond measures 45°. How wide is the pond? (Review 13.6)