n\text{th} \text{ Roots and Rational Exponents}

\textbf{GOAL 1 \hspace{1em} EVALUATING n\text{th} \text{ ROOTS}}

You can extend the concept of a square root to other types of roots. For instance, 2 is a cube root of 8 because \(2^3 = 8\), and 3 is a fourth root of 81 because \(3^4 = 81\). In general, for an integer \(n\) greater than 1, if \(b^n = a\), then \(b\) is an \(n\text{th} \text{ root of} \ a\). An \(n\text{th} \text{ root of} \ a\) is written as \(\sqrt[n]{a}\), where \(n\) is the \textit{index} of the radical.

You can also write an \(n\text{th} \text{ root of} \ a\) as a power of \(a\). For the particular case of a square root, suppose that \(\sqrt[n]{a} = a^{\frac{1}{n}}\). Then you can determine a value for \(k\) as follows:

\[
\sqrt[n]{a} \cdot \sqrt[n]{a} = a
\]
\[
\square \text{ Definition of square root}
\]
\[
a^k \cdot a^k = a
\]
\[
\square \text{ Substitute } a^k \text{ for } \sqrt[n]{a}.
\]
\[
a^{2k} = a^1
\]
\[
\square \text{ Product of powers property}
\]
\[
2k = 1
\]
\[
\square \text{ Set exponents equal when bases are equal.}
\]
\[
k = \frac{1}{2}
\]
\[
\square \text{ Solve for } k.
\]

Therefore, you can see that \(\sqrt[n]{a} = a^{\frac{1}{n}}\). In a similar way you can show that \(\sqrt[n]{a} = a^{\frac{1}{n}}\) and \(\sqrt[n]{a} = a^{\frac{1}{n}}\). In general, \(\sqrt[n]{a} = a^{\frac{1}{n}}\) for any integer \(n\) greater than 1.

\textbf{REAL N^{TH} ROOTS}

Let \(n\) be an integer greater than 1 and let \(a\) be a real number.

- If \(n\) is odd, then \(a\) has one real \(n\text{th} \text{ root}: \sqrt[n]{a} = a^{\frac{1}{n}}\)
- If \(n\) is even and \(a > 0\), then \(a\) has two real \(n\text{th} \text{ roots}: \pm\sqrt[n]{a} = \pm a^{\frac{1}{n}}\)
- If \(n\) is even and \(a = 0\), then \(a\) has one \(n\text{th} \text{ root}: \sqrt[n]{0} = 0^{\frac{1}{n}} = 0\)
- If \(n\) is even and \(a < 0\), then \(a\) has no real \(n\text{th} \text{ roots.}\)

\textbf{EXAMPLE 1 \hspace{1em} Finding \(n\text{th} \text{ Roots}}

Find the indicated real \(n\text{th} \text{ root(s) of} \ a\).

\textbf{a.} \(n = 3\), \(a = -125\)

\textbf{b.} \(n = 4\), \(a = 16\)

\textbf{SOLUTION}

\textbf{a.} Because \(n = 3\) is odd, \(a = -125\) has one real cube root. Because \((-5)^3 = -125\), you can write:

\[
\sqrt[3]{-125} = -5 \quad \text{or} \quad (-125)^{\frac{1}{3}} = -5
\]

\textbf{b.} Because \(n = 4\) is even and \(a = 16 > 0\), 16 has two real fourth roots. Because \(2^4 = 16\) and \((-2)^4 = 16\), you can write:

\[
\pm\sqrt[4]{16} = \pm 2 \quad \text{or} \quad \pm 16^{\frac{1}{4}} = \pm 2
\]
A rational exponent does not have to be of the form \( \frac{1}{n} \) where \( n \) is an integer greater than 1. Other rational numbers such as \( \frac{3}{2} \) and \( -\frac{1}{2} \) can also be used as exponents.

### RATIONAL EXPONENTS

Let \( a^{\frac{1}{n}} \) be an \( n \)th root of \( a \), and let \( m \) be a positive integer.

- \( a^{\frac{m}{n}} = \left( a^{\frac{1}{n}} \right)^m = (\sqrt[n]{a})^m \)
- \( a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{(a^{\frac{1}{n}})^m} = \frac{1}{(\sqrt[n]{a})^m}, \quad a \neq 0 \)

### EXAMPLE 2 Evaluating Expressions with Rational Exponents

\( a.\ 9^{3/2} = (\sqrt{9})^3 = 3^3 = 27 \quad \text{Using radical notation} \)

\( 9^{3/2} = (9^{1/2})^3 = 3^3 = 27 \quad \text{Using rational exponent notation} \)

\( b.\ 32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{2^2} = \frac{1}{4} \quad \text{Using radical notation} \)

\( 32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{(32^{1/5})^2} = \frac{1}{2^2} = \frac{1}{4} \quad \text{Using rational exponent notation} \)

When using a graphing calculator to approximate an \( n \)th root, you may have to rewrite the \( n \)th root using a rational exponent. Then use the calculator’s power key.

### EXAMPLE 3 Approximating a Root with a Calculator

Use a graphing calculator to approximate \( (\sqrt{5})^3 \).

**SOLUTION** First rewrite \( (\sqrt{5})^3 \) as \( 5^{3/4} \). Then enter the following:

**Keystrokes:** 5 \(^{\left(\frac{3}{4}\right)}\) \( \) \( \) \( \) ENTER

**Display:** 3.343701525

\( (\sqrt{5})^3 \approx 3.34 \)

To solve simple equations involving \( x^n \), isolate the power and then take the \( n \)th root of each side.

### EXAMPLE 4 Solving Equations Using \( n \)th Roots

\( a.\ 2x^4 = 162 \)

\[ x^4 = 81 \]

\[ x = \pm \sqrt[4]{81} \]

\[ x = \pm 3 \]

\( b.\ (x - 2)^3 = 10 \)

\[ x - 2 = \sqrt[3]{10} \]

\[ x = \sqrt[3]{10} + 2 \]

\[ x \approx 4.15 \]
### Example 5: Evaluating a Model with nth Roots

The total mass $M$ (in kilograms) of a spacecraft that can be propelled by a magnetic sail is, in theory, given by

$$M = \frac{0.015m^2}{fd^{4/3}}$$

where $m$ is the mass (in kilograms) of the magnetic sail, $f$ is the drag force (in newtons) of the spacecraft, and $d$ is the distance (in astronomical units) to the sun. Find the total mass of a spacecraft that can be sent to Mars using $m = 5000$ kg, $f = 4.52$ N, and $d = 1.52$ AU.  

**Solution**

\[
M = \frac{0.015m^2}{fd^{4/3}}
\]

Write model for total mass.

\[
= \frac{0.015(5000)^2}{4.52(1.52)^{4/3}}
\]

Substitute for $m$, $f$, and $d$.

\[
= 47,500
\]

Use a calculator.

The spacecraft can have a total mass of about 47,500 kilograms. (For comparison, the liftoff weight for a space shuttle is usually about 2,040,000 kilograms.)

### Example 6: Solving an Equation Using an nth Root

**NAUTICAL SCIENCE** The *Olympias* is a reconstruction of a trireme, a type of Greek galley ship used over 2000 years ago. The power $P$ (in kilowatts) needed to propel the *Olympias* at a desired speed $s$ (in knots) can be modeled by this equation:

$$P = 0.0289s^3$$

A volunteer crew of the *Olympias* was able to generate a maximum power of about 10.5 kilowatts. What was their greatest speed?  

**Solution**

\[
P = 0.0289s^3
\]

Write model for power.

\[
10.5 = 0.0289s^3
\]

Substitute 10.5 for $P$.

\[
363 = s^3
\]

Divide each side by 0.0289.

\[
\sqrt[3]{363} = s
\]

Take cube root of each side.

\[
7 = s
\]

Use a calculator.

The greatest speed attained by the *Olympias* was approximately 7 knots (about 8 miles per hour).
GUIDED PRACTICE

Vocabulary Check ✓
Concept Check ✓

1. What is the index of a radical?

2. LOGICAL REASONING Let n be an integer greater than 1. Tell whether the given statement is always true, sometimes true, or never true. Explain.
   a. If \( x^n = a \), then \( x = \sqrt[n]{a} \).
   b. \( a^{1/n} = 1 \).

3. Try to evaluate the expressions \(-\sqrt[4]{625}\) and \(\sqrt[4]{625}\). Explain the difference in your results.

Skill Check ✓

Evaluate the expression.
4. \(\sqrt{81}\)  
5. \(-\sqrt{41/2}\)  
6. \((\sqrt[5]{8})^5\)  
7. \(3125^{2/5}\)

Solve the equation.
8. \(x^3 = 125\)  
9. \(3x^5 = -3\)  
10. \((x + 4)^2 = 0\)  
11. \(x^4 - 7 = 9993\)

12. SHOT PUT The shot (a metal sphere) used in men’s shot put has a volume of about 905 cubic centimeters. Find the radius of the shot. (Hint: Use the formula \(V = \frac{4}{3}\pi r^3\) for the volume of a sphere.)

PRACTICE AND APPLICATIONS

Student Help

Extra Practice to help you master skills is on p. 949.

USING RATIONAL EXPONENT NOTATION Rewrite the expression using rational exponent notation.
13. \(\sqrt[3]{14}\)  
14. \(\sqrt[4]{11}\)  
15. \((\sqrt[5]{5})^2\)  
16. \((\sqrt[6]{16})^5\)  
17. \((\sqrt[2]{2})^{11}\)

USING RADICAL NOTATION Rewrite the expression using radical notation.
18. \(6^{1/3}\)  
19. \(7^{1/4}\)  
20. \(10^{3/7}\)  
21. \(5^{2/5}\)  
22. \(8^{7/4}\)

FINDING NTH ROOTS Find the indicated real nth root(s) of \(a\).
23. \(n = 2, a = 100\)  
24. \(n = 4, a = 0\)  
25. \(n = 3, a = -8\)  
26. \(n = 7, a = 128\)  
27. \(n = 6, a = -1\)  
28. \(n = 5, a = 0\)

EVALUATING EXPRESSIONS Evaluate the expression without using a calculator.
29. \(\sqrt[4]{64}\)  
30. \(\sqrt[3]{1000}\)  
31. \(-\sqrt[6]{64}\)  
32. \(4^{-1/2}\)  
33. \(1^{1/3}\)  
34. \(-256^{1/4}\)  
35. \((\sqrt[4]{16})^2\)  
36. \((\sqrt[4]{-27})^{-4}\)  
37. \((\sqrt[3]{0})^3\)  
38. \(-25^{-3/2}\)  
39. \(32^{4/5}\)  
40. \((-125)^{-2/3}\)

APPROXIMATING ROOTS Evaluate the expression using a calculator. Round the result to two decimal places when appropriate.
41. \(\sqrt[5]{-16,807}\)  
42. \(\sqrt[4]{1124}\)  
43. \(\sqrt[5]{65,536}\)  
44. \(41^{1/10}\)  
45. \(10^{-1/4}\)  
46. \(-1331^{1/3}\)  
47. \((\sqrt[4]{112})^{-4}\)  
48. \((\sqrt[3]{-280})^3\)  
49. \((\sqrt[3]{6})^2\)  
50. \((-190)^{-4/5}\)  
51. \(26^{-3/4}\)  
52. \(522^{2/7}\)
SOLVING EQUATIONS  Solve the equation. Round your answer to two decimal places when appropriate.

53.  \(x^5 = 243\)  
54.  \(6x^3 = -1296\)  
55.  \(x^6 + 10 = 10\)  
56.  \((x - 4)^4 = 81\)  
57.  \(-x^7 = 40\)  
58.  \(-12x^4 = -48\)  
59.  \((x + 12)^3 = 21\)  
60.  \(x^3 - 14 = 22\)  
61.  \(x^8 - 25 = -10\)

62. **BIOLOGY CONNECTION**  For mammals, the lung volume \(V\) (in milliliters) can be modeled by \(V = 170m^{4/5}\) where \(m\) is the body mass (in kilograms). Find the lung volume of each mammal in the table shown.

<table>
<thead>
<tr>
<th>Mammal</th>
<th>Body mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banded mongoose</td>
<td>1.14</td>
</tr>
<tr>
<td>Camel</td>
<td>229</td>
</tr>
<tr>
<td>Horse</td>
<td>510</td>
</tr>
<tr>
<td>Swiss cow</td>
<td>700</td>
</tr>
</tbody>
</table>

63. **SPILLWAY OF A DAM**  A dam’s spillway capacity is an indication of how the dam will perform under certain flood conditions. The spillway capacity \(q\) (in cubic feet per second) of a dam can be calculated using the formula \(q = ch^{3/2}\) where \(c\) is the discharge coefficient, \(\ell\) is the length (in feet) of the spillway, and \(h\) is the height (in feet) of the water on the spillway. A dam with a spillway 40 feet long, 5 feet deep, and 5 feet wide has a discharge coefficient of 2.79. What is the dam’s maximum spillway capacity?

64. **INFLATION**  If the price of an item increases from \(p_1\) to \(p_2\) over a period of \(n\) years, the annual rate of inflation \(i\) (expressed as a decimal) can be modeled by \(i = \left(\frac{p_2}{p_1}\right)^{1/n} - 1.\) In 1940 the average value of a home was $2900. In 1990 the average value was $79,100. What was the rate of inflation for a home?

65. **GEOMETRY CONNECTION**  The formula for the volume \(V\) of a regular dodecahedron (a solid with 12 regular pentagons as faces) is \(V = 7.66a^3\) where \(a\) is the length of an edge of the dodecahedron. Find the length of an edge of a regular dodecahedron that has a volume of 30 cubic feet. Round your answer to two decimal places.

66. **GEOMETRY CONNECTION**  The formula for the volume \(V\) of a regular icosahedron (a solid with 20 congruent equilateral triangles as faces) is \(V = 2.18a^3\) where \(a\) is the length of an edge of the icosahedron. Find the length of an edge of a regular icosahedron that has a volume of 21 cubic centimeters. Round your answer to two decimal places.

67. **ISLAND SPECIES**  Philip Darlington discovered a rule of thumb that relates an island’s land area \(A\) (in square miles) to the number \(s\) of reptile and amphibian species the island can support by the model \(A = 0.0779s^3.\) The area of Puerto Rico is roughly 4000 square miles. About how many reptile and amphibian species can it support?

Source: *The Song of the Dodo: Island Biogeography in an Age of Extinctions*
68. **MULTI-STEP PROBLEM** A board foot is a unit for measuring wood. One board foot has a volume of 144 cubic inches. The Doyle log rule, given by 
\[ b = \left( \frac{r - 2}{2} \right)^2, \]
is a formula for approximating the number \( b \) of board feet in a log with length \( l \) (in feet) and radius \( r \) (in inches). 

The total volume \( V \) (in cubic inches) of wood in the main trunk of a Douglas fir can be modeled by 
\[ V = 250r^3 \]
where \( r \) is the radius of the trunk at the base of the tree. Suppose you need 5000 board feet from a 20 foot Douglas fir log.

a. What volume of wood do you need?

b. What is the radius of a log that will meet your needs?

c. What is the total volume of wood in the main trunk of a Douglas fir tree that will meet your needs?

d. If you find a suitable tree, what fraction of the tree would you actually use?

e. **Writing** How does your answer to part (d) change if you instead need only 2500 board feet?

69. **VISUAL THINKING** Copy the table. Give the number of \( n \)th roots of \( a \) for each category.

<table>
<thead>
<tr>
<th>( n ) is even</th>
<th>( a &lt; 0 )</th>
<th>( a = 0 )</th>
<th>( a &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n ) is odd</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

70. The graph of \( y = x^n \) where \( n \) is even is shown in red. Explain how the graph justifies the table for \( n \) even.

71. Draw a similar graph to justify the table for \( n \) odd.

**MIXED REVIEW**

**SOLVING SYSTEMS** Use Cramer’s rule to solve the linear system. *(Review 4.3)*

72. \( x + 4y = 12 \)
73. \( x - 2y = 11 \)
74. \( 2x - 4y = 7 \)

75. \( 2x + 5y = 18 \)
76. \( 2x + 5y = -14 \)
77. \( -x + y = 1 \)

**SIMPLIFYING EXPRESSIONS** Simplify the expression. Tell which properties of exponents you used. *(Review 6.1 for 7.2)*

78. \( x^4 \cdot x^{-2} \)
79. \( (x^{-3})^5 \)
80. \( (2xy^3)^{-2} \)
81. \( 5x^{-2}y^0 \)

82. \( \frac{x^3}{x^{-4}} \)
83. \( \left( \frac{x^{-2}}{y} \right)^2 \)
84. \( \frac{7x^3y^8}{14xy^{-2}} \)
85. \( \frac{16xy}{9x^5} \cdot \frac{9x^6y}{4y} \)

**FINDING ZEROS** Find all the zeros of the polynomial function. *(Review 6.7)*

86. \( f(x) = x^4 + 9x^3 - 5x^2 - 153x - 140 \)
87. \( f(x) = x^4 + x^3 - 19x^2 + 11x + 30 \)
88. \( f(x) = x^3 - 5x^2 + 16x - 80 \)
89. \( f(x) = x^3 - x^2 + 9x - 9 \)