6.7 Using the Fundamental Theorem of Algebra

**GOAL 1** The Fundamental Theorem of Algebra

The following important theorem, called the fundamental theorem of algebra, was first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855).

**THE FUNDAMENTAL THEOREM OF ALGEBRA**

If \( f(x) \) is a polynomial of degree \( n \) where \( n > 0 \), then the equation \( f(x) = 0 \) has at least one root in the set of complex numbers.

In the following activity you will investigate how the number of solutions of \( f(x) = 0 \) is related to the degree of the polynomial \( f(x) \).

**ACTIVITY** Investigating the Number of Solutions

1. Solve each polynomial equation. State how many solutions the equation has, and classify each as rational, irrational, or imaginary.
   - a. \( 2x - 1 = 0 \)
   - b. \( x^2 - 2 = 0 \)
   - c. \( x^3 - 1 = 0 \)
   
   Make a conjecture about the relationship between the degree of a polynomial \( f(x) \) and the number of solutions of \( f(x) = 0 \).

2. Solve the equation \( x^3 + x^2 - x - 1 = 0 \). How many different solutions are there? How can you reconcile this number with your conjecture?

The equation \( x^3 - 6x^2 - 15x + 100 = 0 \), which can be written as \( (x + 4)(x - 5)^2 = 0 \), has only two distinct solutions: \(-4\) and \(5\). Because the factor \( x - 5 \) appears twice, however, you can count the solution \( 5 \) twice. So, with \( 5 \) counted as a repeated solution, this third-degree equation can be said to have three solutions: \(-4\), \(5\), and \(5\).

In general, when all real and imaginary solutions are counted (with all repeated solutions counted individually), an \( n \)-th degree polynomial equation has exactly \( n \) solutions. Similarly, any \( n \)-th degree polynomial function has exactly \( n \) zeros.

**EXAMPLE 1** Finding the Number of Solutions or Zeros

- a. The equation \( x^3 + 3x^2 + 16x + 48 = 0 \) has three solutions: \(-3\), \(4i\), and \(-4i\).
- b. The function \( f(x) = x^4 + 6x^3 + 12x^2 + 8x \) has four zeros: \(-2\), \(-2\), \(-2\), and \(0\).
**Example 2** Finding the Zeros of a Polynomial Function

Find all the zeros of \( f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6 \).

**Solution**

The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \) and \( \pm 6 \). Using synthetic division, you can determine that \( 1 \) is a repeated zero and that \( -2 \) is also a zero. You can write the function in factored form as follows:

\[
f(x) = (x - 1)(x - 1)(x + 2)(x^2 - 2x + 3)
\]

Complete the factorization, using the quadratic formula to factor the trinomial.

\[
f(x) = (x - 1)(x - 1)(x + 2)[x - (1 + i\sqrt{2})][x - (1 - i\sqrt{2})]
\]

This factorization gives the following five zeros:

\( 1, 1, -2, 1 + i\sqrt{2}, \) and \( 1 - i\sqrt{2} \)

The graph of \( f \) is shown at the right. Note that only the real zeros appear as \( x \)-intercepts. Also note that the graph only touches the \( x \)-axis at the repeated zero \( x = 1 \), but crosses the \( x \)-axis at the zero \( x = -2 \).

The graph in Example 2 illustrates the behavior of the graph of a polynomial function near its zeros. When a factor \( x - k \) is raised to an odd power, the graph crosses the \( x \)-axis at \( x = k \). When a factor \( x - k \) is raised to an even power, the graph is tangent to the \( x \)-axis at \( x = k \).

In Example 2 the zeros \( 1 + i\sqrt{2} \) and \( 1 - i\sqrt{2} \) are complex conjugates. The complex zeros of a polynomial function with real coefficients always occur in complex conjugate pairs. That is, if \( a + bi \) is a zero, then \( a - bi \) must also be a zero.

**Example 3** Using Zeros to Write Polynomial Functions

Write a polynomial function \( f \) of least degree that has real coefficients, a leading coefficient of 1, and 2 and \( 1 + i \) as zeros.

**Solution**

Because the coefficients are real and \( 1 + i \) is a zero, \( 1 - i \) must also be a zero. Use the three zeros and the factor theorem to write \( f(x) \) as a product of three factors.

\[
f(x) = (x - 2)[x - (1 + i)][x - (1 - i)]
\]

Write \( f(x) \) in factored form.

\[
= (x - 2)[(x - 1) - i][(x - 1) + i]
\]

Regroup terms.

\[
= (x - 2)[(x - 1)^2 - i^2]
\]

Multiply.

\[
= (x - 2)[x^2 - 2x + 1 - (-1)]
\]

Expand power and use \( i^2 = -1 \).

\[
= (x - 2)(x^2 - 2x + 2)
\]

Simplify.

\[
= x^3 - 2x^2 + 2x - 2x^2 + 4x - 4
\]

Multiply.

\[
= x^3 - 4x^2 + 6x - 4
\]

Combine like terms.

\( \checkmark \) **Check** You can check this result by evaluating \( f(x) \) at each of its three zeros.
GOAL 2 USING TECHNOLOGY TO APPROXIMATE ZEROS

The rational zero theorem gives you a way to find the rational zeros of a polynomial function with integer coefficients. To find the real zeros of any polynomial function, you may need to use technology.

EXAMPLE 4 Approximating Real Zeros

Approximate the real zeros of \( f(x) = x^4 - 2x^3 - x^2 - 2x - 2 \).

**SOLUTION**

There are several ways to use a graphing calculator to approximate the real zeros of a function. One way is to use the Zero (or Root) feature as shown below.

From these screens, you can see that the real zeros are about \(-0.73\) and \(2.73\). Because the polynomial function has degree 4, you know that there must be two other zeros. These may be repeats of the real zeros, or they may be imaginary zeros. In this particular case, the two other zeros are imaginary: \(x = \pm i\).

EXAMPLE 5 Approximating Real Zeros of a Real-Life Function

**PHYSIOLOGY** For one group of people it was found that a person’s score \( S \) on the Harvard Step Test was related to his or her amount of hemoglobin \( x \) (in grams per 100 milliliters of blood) by the following model:

\[
S = -0.015x^3 + 0.6x^2 - 2.4x + 19
\]

The normal range of hemoglobin is 12–18 grams per 100 milliliters of blood. Approximate the amount of hemoglobin for a person who scored 75.

**SOLUTION**

You can solve the equation

\[
75 = -0.015x^3 + 0.6x^2 - 2.4x + 19
\]

by rewriting it as \( 0 = -0.015x^3 + 0.6x^2 - 2.4x - 56 \) and then using a graphing calculator to approximate the real zeros of \( f(x) = -0.015x^3 + 0.6x^2 - 2.4x - 56 \). From the graph you can see that there are three real zeros: \( x \approx -7.3, x \approx 16.4, \) and \( x \approx 30.9 \).

The person’s hemoglobin is probably about 16.4 grams per 100 milliliters of blood, since this is the only zero within the normal range.
1. State the fundamental theorem of algebra.

2. Two zeros of \( f(x) = x^3 - 6x^2 - 16x + 96 \) are 4 and -4. Explain why the third zero must also be a real number.

3. The graph of \( f(x) = x^3 - x^2 - 8x + 12 \) is shown at the right. How many real zeros does the function have? How many imaginary zeros does the function have? Explain your reasoning.

4. \( f(x) = x^3 - x^2 - 2x \)

5. \( f(x) = x^4 + x^2 - 12 \)

6. \( f(x) = x^3 + 5x^2 - 9x - 45 \)

7. \( f(x) = x^4 - 3x^2 - 2x - 3 \)

Find all the zeros of the polynomial function.

8. \( f(x) = x^3 - 4x^2 + 16x - 64, x = 4 \)

9. \( f(x) = x^3 - 3x^2 + x - 3, x = -i \)

Write a polynomial function of least degree that has real coefficients, the given zeros, and a leading coefficient of 1.

10. \( 3, 0, -2 \)

11. \( 1, -1, 2, -2, 3 \)

12. \( 5, 2 + 3i \)

13. \( 1, -1, 2, -2, 3 \)

14. \( 3i, 4 \)

15. \( 4 \)

For the 25 years that a grocery store has been open, its annual revenue \( R \) (in millions of dollars) can be modeled by

\[
R = \frac{1}{10,000}(-t^4 + 12t^3 - 77t^2 + 600t + 13,650)
\]

where \( t \) is the number of years the store has been open. In what year(s) was the revenue $1.5 million?

16. \( f(x) = x^3 + 3x^2 - 5x + 8, x = 4 \)

17. \( f(x) = x^3 + 5x^2 + x + 5, x = -5 \)

18. \( f(x) = x^3 + 5x^2 + x + 5, x = -i \)

19. \( f(x) = x^3 - 4x^2 + 16x - 64, x = 4i \)

20. \( f(x) = x^3 - 3x^2 + x - 3, x = -i \)

CHECKING ZEROS  Decide whether the given \( x \)-value is a zero of the function.

21. \( f(x) = x^3 + 5x^2 + 5x^2 - 5x - 6 \)

22. \( f(x) = x^4 + 4x^3 - 6x^2 - 36x - 27 \)

23. \( f(x) = x^3 - 4x^2 + 3x \)

24. \( f(x) = x^3 + 5x^2 - 4x - 20 \)

25. \( f(x) = x^4 + 7x^3 - x^2 - 67x - 60 \)

26. \( f(x) = x^4 - 5x^2 - 36 \)

27. \( f(x) = x^3 - x^2 + 49x - 49 \)

28. \( f(x) = x^3 - x^2 + 25x - 25 \)

29. \( f(x) = x^4 + 6x^3 + 14x^2 + 54x + 45 \)

30. \( f(x) = x^3 + 3x^2 + 25x + 75 \)

31. \( f(x) = x^4 - x^3 - 5x^2 - x - 6 \)

32. \( f(x) = x^4 + x^3 + 2x^2 + 4x - 8 \)

33. \( f(x) = 2x^4 - 7x^3 - 27x^2 + 63x + 81 \)

34. \( f(x) = 2x^4 - x^3 - 42x^2 + 16x + 160 \)

FINDING ZEROS  Find all the zeros of the polynomial function.

35. \( f(x) = x^3 - 2x^2 + 3x - 6 \)

36. \( f(x) = x^3 - 5x^2 + 2x - 8 \)

37. \( f(x) = x^4 - 4x^3 + 3x^2 - 4x + 3 \)

38. \( f(x) = x^4 - 6x^3 + 10x^2 - 6x + 2 \)

39. \( f(x) = x^5 - 3x^4 + 3x^3 - x^2 - x - 1 \)

40. \( f(x) = x^5 - 4x^4 + 6x^3 - 4x^2 + x \)
**Writing Polynomial Functions**

Write a polynomial function of least degree that has real coefficients, the given zeros, and a leading coefficient of 1.

35. 2, 1, 4
36. 1, -4, 5
37. -6, 3, 5
38. -5, 2, -2
39. -2, -4, -7
40. 8, -i, i
41. 3i, -3i, 5
42. 2, -2, -6i
43. i, -3i, 3i
44. 3- i, 5
45. 4, 4, 2 + i
46. -2, -2, 3, -4i

**Finding Zeros**

Use a graphing calculator to graph the polynomial function. Then use the Zero (or Root) feature of the calculator to find the real zeros of the function.

47. \(f(x) = x^3 - x^2 - 5x + 3\)
48. \(f(x) = 2x^3 - x^2 - 3x - 1\)
49. \(f(x) = x^3 - 2x^2 + x + 1\)
50. \(f(x) = x^4 - 2x - 1\)
51. \(f(x) = x^4 - x^3 - 4x^2 - 3x - 2\)
52. \(f(x) = x^4 - x^3 - 3x^2 - x + 1\)
53. \(f(x) = x^4 + 3x^2 - 2\)
54. \(f(x) = x^4 - x^3 - 20x^2 + 10x + 27\)

**Graphing Models**

In Exercises 55–59, you may find it helpful to graph the model on a graphing calculator.

55. **United States Exports**

For 1980 through 1996, the total exports \(E\) (in billions of dollars) of the United States can be modeled by

\[ E = -0.131t^3 + 5.033t^2 - 23.2t + 233 \]

where \(t\) is the number of years since 1980. In what year were the total exports about $312.76 billion? ★ Source: U.S. Bureau of the Census

56. **Education Donations**

For 1983 through 1995, the amount of private donations \(D\) (in millions of dollars) allocated to education can be modeled by

\[ D = 1.78t^3 - 6.02t^2 + 752t + 6701 \]

where \(t\) is the number of years since 1983. In what year was $14.3 billion of private donations allocated to education? ★ Source: AAFRC Trust for Philanthropy

57. **Sports Equipment**

For 1987 through 1996, the sales \(S\) (in millions of dollars) of gym shoes and sneakers can be modeled by

\[ S = -0.982t^5 + 24.6t^4 - 211t^3 + 661t^2 - 318t + 1520 \]

where \(t\) is the number of years since 1987. Were there any years in which sales were about $2 billion? Explain. ★ Source: National Sporting Goods Association

58. **Television**

For 1990 through 2000, the actual and projected amount spent on television per person per year in the United States can be modeled by

\[ S = -0.213t^3 + 3.96t^2 + 10.2t + 366 \]

where \(S\) is the amount spent (in dollars) and \(t\) is the number of years since 1990. During which year was $455 spent per person on television?

★ Source: Veronis, Suhler & Associates, Inc.

59. **Population**

For 1890 through 1990, the American Indian, Eskimo, and Aleut population \(P\) (in thousands) can be modeled by the function

\[ P = 0.00496t^3 - 0.432t^2 + 11.3t + 212 \]

where \(t\) is the number of years since 1890. In what year did the population reach 722,000? ★ Data Update of Statistical Abstract of the United States data at www.mcdougallittell.com

**Focus On Applications**

**United States Exports**

The United States exports more than any other country in the world. It also imports more than any other country.
60. **MULTI-STEP PROBLEM** Mary plans to save $1000 each summer to buy a used car at the end of the fourth summer. At the end of each summer, she will deposit the $1000 she earned from her summer job into her bank account. The table shows the value of her deposits over the four year period. In the table, $g$ is the growth factor $1 + r$ where $r$ is the annual interest rate expressed as a decimal.

<table>
<thead>
<tr>
<th>Function</th>
<th>Zeros</th>
<th>Sum of zeros</th>
<th>Product of zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x^2 - 5x + 6$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$f(x) = x^3 - 7x + 6$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$f(x) = x^4 + 2x^3 + x^2 + 8x - 12$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$f(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

b. Use your completed table to make a conjecture relating the sum of the zeros of a polynomial function with the coefficients of the polynomial function.

c. Use your completed table to make a conjecture relating the product of the zeros of a polynomial function with the coefficients of the polynomial function.

61. a. Copy and complete the table.

<table>
<thead>
<tr>
<th>Function</th>
<th>Zeros</th>
<th>Sum of zeros</th>
<th>Product of zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x^2 - 5x + 6$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$f(x) = x^3 - 7x + 6$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$f(x) = x^4 + 2x^3 + x^2 + 8x - 12$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>$f(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

b. Use your completed table to make a conjecture relating the sum of the zeros of a polynomial function with the coefficients of the polynomial function.

c. Use your completed table to make a conjecture relating the product of the zeros of a polynomial function with the coefficients of the polynomial function.

62. Show that the sum of a pair of complex conjugates is a real number.

63. Show that the product of a pair of complex conjugates is a real number.

**MIXED REVIEW**

**GRAPHING WITH INTERCEPT FORM** Graph the quadratic function. Label the vertex, axis of symmetry, and $x$-intercepts. (Review 5.1 for 6.8)

64. $y = -3(x - 2)(x + 2)$
65. $y = 2(x - 1)(x - 5)$
66. $y = 2(x + 4)(x - 3)$
67. $y = -(x + 1)(x - 5)$

**GRAPHING POLYNOMIALS** Graph the polynomial function. (Review 6.2 for 6.8)

68. $f(x) = -2x^4$
69. $f(x) = -x^3 - 4$
70. $f(x) = x^3 + 4x - 3$
71. $f(x) = x^4 - 3x^3 + x + 2$