5.7 Graphing and Solving Quadratic Inequalities

***What** you should learn*

GOAL 1 Graph quadratic inequalities in two variables.

GOAL 2 Solve quadratic inequalities in one variable, as applied in **Example 7**.

***Why** you should learn it*

▼ To solve **real-life** problems, such as finding the weight of theater equipment that a rope can support in **Exs. 47 and 48**.

REAL LIFE

GOAL 1 QUADRATIC INEQUALITIES IN TWO VARIABLES

In this lesson you will study four types of **quadratic inequalities in two variables**.

$$y < ax^2 + bx + c \qquad y \le ax^2 + bx + c$$

$$y > ax^2 + bx + c \qquad y \ge ax^2 + bx + c$$

The graph of any such inequality consists of all solutions (x, y) of the inequality. The steps used to graph a quadratic inequality are very much like those used to graph a linear inequality. (See Lesson 2.6.)

GRAPHING A QUADRATIC INEQUALITY IN TWO VARIABLES

To graph one of the four types of quadratic inequalities shown above, follow these steps:

STEP 1 Draw the parabola with equation $y = ax^2 + bx + c$. Make the parabola *dashed* for inequalities with $<$ or $>$ and *solid* for inequalities with $\le$ or $\ge$.

STEP 2 Choose a point (x, y) inside the parabola and check whether the point is a solution of the inequality.

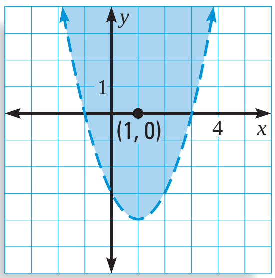
STEP 3 If the point from Step 2 is a solution, shade the region inside the parabola. If it is not a solution, shade the region outside the parabola.

EXAMPLE 1 *Graphing a Quadratic Inequality*

Graph $y > x^2 - 2x - 3$.

SOLUTION

Follow Steps 1–3 listed above.



1. Graph $y = x^2 - 2x - 3$. Since the inequality symbol is $>$, make the parabola dashed.

2. Test a point inside the parabola, such as $(1, 0)$.

$$y > x^2 - 2x - 3$$

$$0 \overset{?}{>} 1^2 - 2(1) - 3$$

$$0 > -4 \checkmark$$

So, $(1, 0)$ is a solution of the inequality.

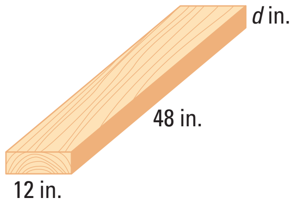
3. Shade the region inside the parabola.

REAL LIFE
Carpentry

EXAMPLE 2 *Using a Quadratic Inequality as a Model*

You are building a wooden bookcase. You want to choose a thickness d (in inches) for the shelves so that each is strong enough to support 60 pounds of books without breaking. A shelf can safely support a weight of W (in pounds) provided that:

$$W \le 300d^2$$



a. Graph the given inequality.

b. If you make each shelf 0.75 inch thick, can it support a weight of 60 pounds?

SOLUTION

STUDENT HELP

Look Back
For help with graphing inequalities in two variables, see p. 108.

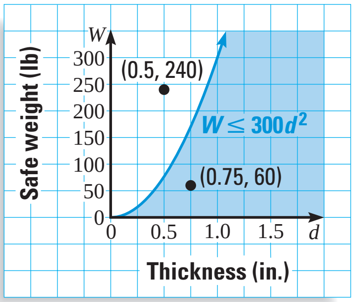
a. Graph $W = 300d^2$ for nonnegative values of d. Since the inequality symbol is $\le$, make the parabola solid. Test a point inside the parabola, such as (0.5, 240).

$$W \le 300d^2$$

$$240 \overset{?}{\le} 300(0.5)^2$$

$$240 \not\le 75$$

Since the chosen point is not a solution, shade the region outside (below) the parabola.



b. The point (0.75, 60) lies in the shaded region of the graph from part (a), so (0.75, 60) is a solution of the given inequality. Therefore, a shelf that is 0.75 inch thick *can* support a weight of 60 pounds.

. . . .

Graphing a *system* of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the system. Then identify the region in the coordinate plane common to all the graphs. This region is called the *graph of the system.*

EXAMPLE 3 *Graphing a System of Quadratic Inequalities*

Graph the system of quadratic inequalities.

$y \ge x^2 - 4$ **Inequality 1**

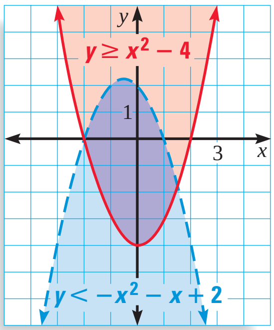
$y < -x^2 - x + 2$ **Inequality 2**

SOLUTION

Graph the inequality $y \ge x^2 - 4$. The graph is the red region inside and including the parabola $y = x^2 - 4$.

Graph the inequality $y < -x^2 - x + 2$. The graph is the blue region inside (but not including) the parabola $y = -x^2 - x + 2$.

Identify the **purple region** where the two graphs overlap. This region is the graph of the system.



GOAL 2 QUADRATIC INEQUALITIES IN ONE VARIABLE

One way to solve a **quadratic inequality in one variable** is to use a graph.

- To solve $ax^2 + bx + c < 0$ (or $ax^2 + bx + c \le 0$), graph $y = ax^2 + bx + c$ and identify the x-values for which the graph lies *below* (or *on and below*) the x-axis.
- To solve $ax^2 + bx + c > 0$ (or $ax^2 + bx + c \ge 0$), graph $y = ax^2 + bx + c$ and identify the x-values for which the graph lies *above* (or *on and above*) the x-axis.

EXAMPLE 4 *Solving a Quadratic Inequality by Graphing*

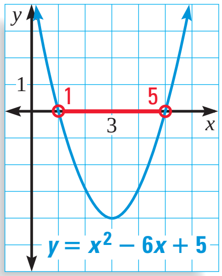
STUDENT HELP

Look Back
For help with solving inequalities in one variable, see p. 41.

Solve $x^2 - 6x + 5 < 0$.

SOLUTION

The solution consists of the x-values for which the graph of $y = x^2 - 6x + 5$ lies below the x-axis. Find the graph's x-intercepts by letting $y = 0$ and using factoring to solve for x.



$$0 = x^2 - 6x + 5$$

$$0 = (x - 1)(x - 5)$$

$$x = 1 \text{ or } x = 5$$

Sketch a parabola that opens up and has 1 and 5 as x-intercepts. The graph lies below the x-axis between $x = 1$ and $x = 5$.

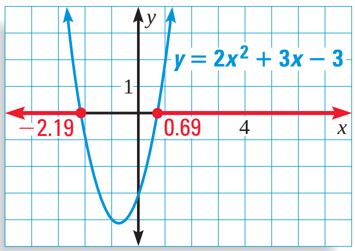
▶ The solution of the given inequality is $1 < x < 5$.

EXAMPLE 5 *Solving a Quadratic Inequality by Graphing*

Solve $2x^2 + 3x - 3 \ge 0$.

SOLUTION

The solution consists of the x-values for which the graph of $y = 2x^2 + 3x - 3$ lies on and above the x-axis. Find the graph's x-intercepts by letting $y = 0$ and using the quadratic formula to solve for x.



$$0 = 2x^2 + 3x - 3$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{33}}{4}$$

$$x \approx 0.69 \text{ or } x \approx -2.19$$

Sketch a parabola that opens up and has 0.69 and -2.19 as x-intercepts. The graph lies on and above the x-axis to the left of (and including) $x = -2.19$ and to the right of (and including) $x = 0.69$.

▶ The solution of the given inequality is approximately $x \le -2.19$ or $x \ge 0.69$.

You can also use an algebraic approach to solve a quadratic inequality in one variable, as demonstrated in Example 6.

EXAMPLE 6 *Solving a Quadratic Inequality Algebraically*

Solve $x^2 + 2x \le 8$.

SOLUTION

First write and solve the equation obtained by replacing the inequality symbol with an equals sign.

$x^2 + 2x \le 8$ **Write original inequality.**

$x^2 + 2x = 8$ **Write corresponding equation.**

$x^2 + 2x - 8 = 0$ **Write in standard form.**

$(x + 4)(x - 2) = 0$ **Factor.**

$x = -4$ or $x = 2$ **Zero product property**

The numbers -4 and 2 are called the *critical x-values* of the inequality $x^2 + 2x \le 8$. Plot -4 and 2 on a number line, using solid dots because the values satisfy the inequality. The critical x-values partition the number line into three intervals. Test an x-value in each interval to see if it satisfies the inequality.

Test $x = -5$: $(-5)^2 + 2(-5) = 15 \not\le 8$

Test $x = 0$: $0^2 + 2(0) = 0 \le 8$ ✓

Test $x = 3$: $3^2 + 2(3) = 15 \not\le 8$

▶ The solution is $-4 \le x \le 2$.

EXAMPLE 7 *Using a Quadratic Inequality as a Model*

DRIVING For a driver aged x years, a study found that the driver's reaction time $V(x)$ (in milliseconds) to a visual stimulus such as a traffic light can be modeled by:

$$V(x) = 0.005x^2 - 0.23x + 22, \quad 16 \le x \le 70$$

At what ages does a driver's reaction time tend to be greater than 25 milliseconds?

▶ Source: *Science Probe!*

FOCUS ON APPLICATIONS

DRIVING Driving simulators help drivers safely improve their reaction times to hazardous situations they may encounter on the road.

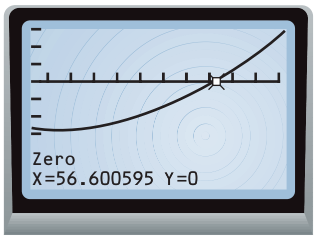
SOLUTION

You want to find the values of x for which:

$$V(x) > 25$$

$$0.005x^2 - 0.23x + 22 > 25$$

$$0.005x^2 - 0.23x - 3 > 0$$



Graph $y = 0.005x^2 - 0.23x - 3$ on the domain $16 \le x \le 70$. The graph's x-intercept is about 57, and the graph lies above the x-axis when $57 < x \le 70$.

▶ Drivers over 57 years old tend to have reaction times greater than 25 milliseconds.

Guided Practice

Vocabulary Check ✓

1. Give one example each of a quadratic inequality in one variable and a quadratic inequality in two variables.

Concept Check ✓

2. How does the graph of $y > x^2$ differ from the graph of $y \geq x^2$?
3. Explain how to solve $x^2 - 3x - 4 > 0$ graphically and algebraically.

Skill Check ✓

Graph the inequality.

4. $y \geq x^2 + 2$
5. $y \leq -2x^2$
6. $y < x^2 - 5x + 4$

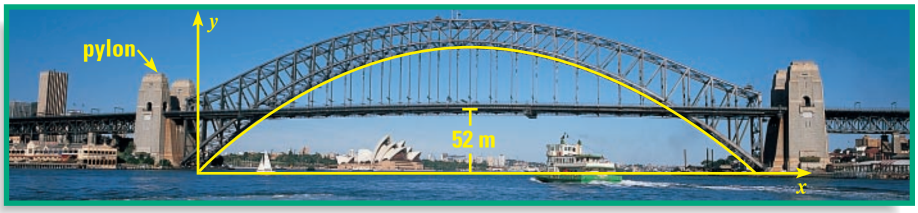
Graph the system of inequalities.

7. $y \leq -x^2 + 3$, $y \geq x^2 + 2x - 4$
8. $y \geq -x^2 + 3$, $y \geq x^2 + 2x - 4$
9. $y \geq -x^2 + 3$, $y \leq x^2 + 2x - 4$

Solve the inequality.

10. $x^2 - 4 < 0$
11. $x^2 - 4 \geq 0$
12. $x^2 - 4 > 3x$

13. **ARCHITECTURE** The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by $y = -0.00211x^2 + 1.06x$ where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. For what distances x is the arch above the road?



Practice and Applications

STUDENT HELP

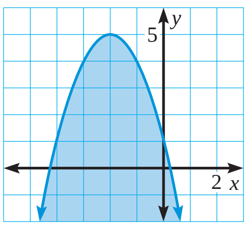
Extra Practice to help you master skills is on p. 947.

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 14–28
Example 2: Exs. 47–49
Example 3: Exs. 29–34, 49
Examples 4, 5: Exs. 35–40
Example 6: Exs. 41–46
Example 7: Exs. 50, 51

MATCHING GRAPHS **Match the inequality with its graph.**

14. $y \geq x^2 - 4x + 1$
15. $y < x^2 - 4x + 1$
16. $y \leq -x^2 - 4x + 1$

A.


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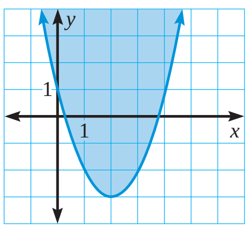
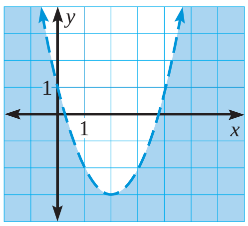


C.


GRAPHING QUADRATIC INEQUALITIES **Graph the inequality.**

17. $y \geq 3x^2$
18. $y \leq -x^2$
19. $y > -x^2 + 5$
20. $y < x^2 - 3x$
21. $y \leq x^2 + 8x + 16$
22. $y \leq -x^2 + x + 6$
23. $y \geq 2x^2 - 2x - 5$
24. $y \geq -2x^2 - x + 3$
25. $y > -3x^2 + 5x - 4$
26. $y < -\frac{1}{2}x^2 - 2x + 4$
27. $y > \frac{4}{3}x^2 - 12x + 29$
28. $y < 0.6x^2 + 3x + 2.4$

FOCUS ON CAREERS

SET DESIGNER
A set designer creates the scenery, or *sets*, used in a theater production. The designer may make scale models of the sets before they are actually built.

CAREER LINK
www.mcdougallittell.com

GRAPHING SYSTEMS Graph the system of inequalities.

29. $y \geq x^2$
$y \leq x^2 + 3$

30. $y < -3x^2$
$y \geq -\frac{1}{2}x^2 - 5$

31. $y > x^2 - 6x + 9$
$y < -x^2 + 6x - 3$

32. $y \geq x^2 + 2x + 1$
$y \geq x^2 - 4x + 4$

33. $y < 3x^2 + 2x - 5$
$y \geq -2x^2 + 1$

34. $y \leq 2x^2 - 9x + 8$
$y > -x^2 - 6x - 4$

SOLVING BY GRAPHING Solve the inequality by graphing.

35. $x^2 + x - 2 < 0$

36. $2x^2 - 7x + 3 \geq 0$

37. $-x^2 - 2x + 8 \leq 0$

38. $-x^2 + x + 5 > 0$

39. $3x^2 + 24x \geq -41$

40. $-\frac{3}{4}x^2 + 4x - 8 < 0$

SOLVING ALGEBRAICALLY Solve the inequality algebraically.

41. $x^2 + 3x - 18 \geq 0$

42. $3x^2 - 16x + 5 \leq 0$

43. $4x^2 < 25$

44. $-x^2 - 12x < 32$

45. $2x^2 - 4x - 5 > 0$

46. $\frac{1}{2}x^2 + 3x \leq -6$

THEATER **In Exercises 47 and 48, use the following information.**
You are a member of a theater production crew. You use manila rope and wire rope to support lighting, scaffolding, and other equipment. The weight W (in pounds) that can be safely supported by a rope with diameter d (in inches) is given below for both types of rope. ▶ Source: *Workshop Math*

Manila rope: $W \leq 1480d^2$ **Wire rope:** $W \leq 8000d^2$

47. Graph the inequalities in separate coordinate planes for $0 \leq d \leq 1\frac{1}{2}$.

48. Based on your graphs, can 1000 pounds of theater equipment be supported by a $\frac{1}{2}$ inch manila rope? by a $\frac{1}{2}$ inch wire rope?

49. **HEALTH** For a person of height h (in inches), a healthy weight W (in pounds) is one that satisfies this system of inequalities:

$$W \geq \frac{19h^2}{703} \quad \text{and} \quad W \leq \frac{25h^2}{703}$$

Graph the system for $0 \leq h \leq 80$. What is the range of healthy weights for a person 67 inches tall? ▶ Source: *Parade Magazine*

SOLVING INEQUALITIES **In Exercises 50–52, you may want to use a graphing calculator to help you solve the problems.**

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 50–52.

50. **FORESTRY** *Sawtimber* is a term for trees that are suitable for sawing into lumber, plywood, and other products. For the years 1983–1995, the unit value y (in 1994 dollars per million board feet) of one type of sawtimber harvested in California can be modeled by

$$y = 0.125x^2 - 569x + 848{,}000, \quad 400 \leq x \leq 2200$$

where x is the volume of timber harvested (in millions of board feet).
▶ Source: California Department of Forestry and Fire Protection

a. For what harvested timber volumes is the value of the timber at least \$400,000 per million board feet?

b. **LOGICAL REASONING** What happens to the unit value of the timber as the volume harvested increases? Why would you expect this to happen?

51. **MEDICINE** In 1992 the average income I (in dollars) for a doctor aged x years could be modeled by:

$$I = -425x^2 + 42{,}500x - 761{,}000$$

For what ages did the average income for a doctor exceed $250,000?

DATA UPDATE of *American Almanac of Jobs and Salaries* data at www.mcdougallittell.com

Test Preparation

52. **MULTI-STEP PROBLEM** A study of driver reaction times to audio stimuli found that the reaction time $A(x)$ (in milliseconds) of a driver can be modeled by

$$A(x) = 0.0051x^2 - 0.319x + 15, \quad 16 \le x \le 70$$

where x is the driver's age (in years). ▶ Source: *Science Probe!*

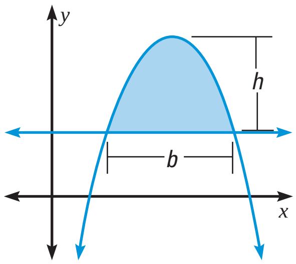
a. Graph $y = A(x)$ on the given domain. Also graph $y = V(x)$, the reaction-time model for visual stimuli from Example 7, in the same coordinate plane.

b. For what values of x in the interval $16 \le x \le 70$ is $A(x) < V(x)$?

c. *Writing* Based on your results from part (b), do you think a driver would react more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.

★ **Challenge**

53. **GEOMETRY CONNECTION** The area A of the region bounded by a parabola and a horizontal line is given by $A = \frac{2}{3}bh$ where b and h are as defined in the diagram. Find the area of the region determined by each pair of inequalities.



a. $y \le -x^2 + 4x$
$y \ge 0$

b. $y \ge x^2 - 4x - 5$
$y \le 3$

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING FOR A VARIABLE **Solve the equation for y.** (Review 1.4)

54. $3x + y = 1$ **55.** $8x - 2y = 10$ **56.** $-2x + 5y = 9$

57. $\frac{1}{6}x + \frac{1}{3}y = -\frac{11}{12}$ **58.** $xy - x = 2$ **59.** $\frac{x - 3y}{4} = 7x$

SOLVING SYSTEMS **Solve the system of linear equations.** (Review 3.6 for 5.8)

60. $5x - 3y - 2z = -17$
$-x + 7y - 3z = 6$
$3x + 2y + 4z = 13$

61. $x - 4y + z = -14$
$2x + 3y + 7z = -15$
$-3x + 5y - 5z = 29$

COMPLEX NUMBERS **Write the expression as a complex number in standard form.** (Review 5.4)

62. $(3 + 4i) + (10 - i)$ **63.** $(-11 - 2i) + (5 + 2i)$

64. $(9 + i) - (4 - i)$ **65.** $(5 - 3i) - (-1 + 2i)$

66. $6i(8 + i)$ **67.** $(7 + 3i)(2 - 5i)$

68. $\frac{1}{3 - i}$ **69.** $\frac{4 - 3i}{9 + 2i}$