

Challenge: Skills and Applications

For use with pages 114–120

1. Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$$

- a. Find each of the following: $f(0)$, $f(\frac{1}{3})$, $f(\frac{1}{2})$, $f(\sqrt{2})$, $f(2)$, $f(\pi)$

- b. Sketch the graph of $y = f(x)$.

2. What is perhaps the ultimate piecewise-defined function was proposed by the German mathematician Gustav Peter Lejeune Dirichlet (1805–1859). It is defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

- a. Find each of the following values: $f(2)$, $f(\frac{1}{2})$, $f(\sqrt{2})$, $f(\frac{5}{4})$, $f(\pi)$.

- b. Explain how you know that Dirichlet's function is actually a function, and not just a relation. Could a computer draw an accurate graph of this function?

3. Suppose m and k are numbers and let $g(x)$ be defined piecewise as follows:

$$g(x) = \begin{cases} \frac{1}{2}x + k & \text{if } x < 3 \\ mx + 2 & \text{if } x \geq 3 \end{cases}$$

- a. Suppose $m = \frac{3}{2}$. What must the value of k be in order for the graph of $y = g(x)$ to be connected?

- b. Suppose $k = 4$. What must the value of m be in order for the graph to be connected?

4. Let $f(x)$ be defined for all nonnegative real numbers as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ f(0)x + f(0) & \text{if } 0 < x \leq 1 \\ f(1)(x - 1) + f(1) & \text{if } 1 < x \leq 2 \\ f(2)(x - 2) + f(2) & \text{if } 2 < x \leq 3 \\ \dots & \end{cases}$$

- a. Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

- b. Sketch the graph of $f(x)$ for $0 \leq x \leq 4$.